Transport in Porous Media **27:** 289–304, 1997.

An Analytical Study of Free Convective Boundary Layer Flow in Porous Media: The Effect of Anisotropic Diffusivity

L. STORESLETTEN[1] and D. A. S. REES[2]
[1]*Department of Mathematics, Agder College, Kristiansand, Norway*
e-mail: leiv.storesletten@hia.no
[2]*School of Mechanical Engineering, University of Bath, Claverton Down, Bath BA2 7AY, U.K.*
e-mail: ensdasr@bath.ac.uk

(Received: 13 November 1995; in final form: 17 February 1997)

Abstract. The effect of an anisotropic thermal diffusivity tensor on the free convective boundary layer flow in porous media is studied. Convection is induced by a generally inclined, uniformly heated surface embedded in a fluid-saturated medium. A third-order boundary layer theory is presented in order to obtain accurate information on the effect of anisotropy on the rate of heat transfer into the porous medium. It is shown that the thickness of the resulting leading order boundary layer flow depends on the precise nature of the anisotropy. On the other hand, the anisotropic diffusivity does not induce a fluid drift in the spanwise direction, a result which is different from that obtained in our earlier study of the effects of an anisotropic permeability. It is found that the second order temperature field does not contribute to the overall rate of heat transfer. Finally, we show that the third-order correction to the leading-order rate of heat transfer is given in terms of an explicit formula.

Key words: free convection, boundary layer, anisotropy.

Nomenclature

b	coefficient of thermal expansion
$\mathbf{D}$	thermal diffusivity tensor
$\mathbf{E}$	nondimensional thermal diffusivity tensor
F	reduced streamfunction
$\mathcal{F}$	reduced streamfunction
g	acceleration due to gravity
G	temperature
$\mathcal{G}$	temperature
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	right-handed set of unit vectors
K	permeability
$\mathcal{L}$	differential operator
p	pressure
r	radial coordinate
$\mathcal{R}$	right hand side of certain equations
T	dimensional temperature
u, v, w	fluid velocities in the x, y and z directions, respectively

$\mathbf{u} = (u, v, w)$	velocity vector
x, y, z	streamwise, cross-stream and spanwise Cartesian coordinates

Greek Symbols

α	angle of rotation about the x-axis
β	angle of rotation about the y-axis
γ	angle of rotation about the z-axis
δ	inclination of the heated surface from the vertical
η	similarity variable
ζ	scaled similarity variable
λ	eigenvalue
μ	dynamic viscosity
ρ	density
ϕ	angular coordinate
ψ	streamfunction
θ	nondimensional temperature
ξ	dummy variable
ω	wedge angle

Superscripts

$'$	differentiation with respect to the appropriate independent variable
$\hat{}$	principal axes of the diffusivity tensor

Subscripts

a	adjoint solution
w	condition at the wall
x, y, z	differentiation
∞	condition at infinity

1. Introduction

Thermally driven convection in porous media is of importance in a variety of geophysical and technological problems including the modelling of geothermal wells and insulation and filtration systems. Work on the theory and modelling of porous media flows began nearly 50 years ago, and has now become an important topic within the general field of fluid dynamics. So far, the great majority of theoretical and experimental investigations have dealt with isotropic media. However, in many practical problems, the porous matrix can be *anisotropic* in its mechanical and/or thermal properties. In this paper we concentrate on the thermal properties by studying the effect of an anisotropic thermal diffusivity. This work is a companion paper to an earlier study by the same authors on the effects of an anisotropic permeability tensor, Rees and Storesletten (1995).

Tyvand and Storesletten (1991) seem to be the first to have studied natural convection where at least one of the principal axes of the permeability tensor is neither parallel with nor perpendicular to a bounding surface. They studied convection in a horizontal porous layer where the anisotropic permeability is transversely isotropic but the orientation of the longitudinal axis is arbitrary. This was sufficient to achieve qualitatively new flow patterns with tilted planes of motion or tilted as well as curved cell walls; this depends, respectively, on whether the transverse permeability is larger or smaller than the longitudinal permeability. Storesletten (1993) has studied the analogous problem for a horizontal layer with anisotropy in the thermal diffusivity. There are again two different types of convection cells depending on whether the transverse diffusivity is smaller or larger than the longitudinal diffusivity. In the former case the convection cells have a rectangular cross-section with vertical lateral walls just like isotropic convection. In the latter case, however, the lateral cell walls are tilted as well as curved. Thus far, studies of natural convection in porous layers indicate that the effect of anisotropy in either the mechanical or the thermal properties of the medium has a much greater influence on the resulting convection pattern when none of the principal axes is normal to the layer. Other recent papers on this general topic include those by Ni and Beckermann (1991) and Degan, Vasseur and Bilgen (1995).

The first papers to appear dealing with thermal boundary layer flow in an isotropic porous medium were by Cheng and Chang (1976) and Cheng and Minkowycz (1977). In these papers, certain geothermal formations are modelled by assuming that they are represented adequately by semi-infinite surfaces which are horizontal and vertical, respectively. Cheng and co-workers assumed further that the boundary layer approximation is valid and analysed the flow and heat transfer by determining the leading-order boundary layer flow. Later, this work was extended to higher-order by Chang and Cheng (1983) , Daniels and Simpkins (1984), Cheng and Hsu (1984) and Riley and Rees (1985). Using the method of matched asymptotic expansions, these authors were able to obtain more accurate accounts of the rate of heat transfer into the porous medium.

In common with all the authors quoted in the above paragraph we shall assume that the heated surface is maintained at a constant, steady temperature. However, we relax the assumption that the porous medium is thermally isotropic. Anisotropy arises naturally in porous media and such media usually exhibit both mechanical and thermal anisotropy. The detailed effects of an anisotropic permeability were dealt with in Rees and Storesletten (1995), whereas the presence of thermal anisotropy in a mechanically isotropic medium is considered here. Although such a medium may be manufactured by the suitable insertion of metallic threads into the matrix (Kvernvold and Tyvand, 1979), we presume that they may not be common in nature. The chief reason for considering solely thermal anisotropy is that very much more analytical progress may be made compared with the general case.

To date, there exists only two papers to our knowledge which consider the effect of anisotropy on thermal boundary layer flow in porous media: Ene (1991) and Rees and Storesletten (1995). In Rees and Storesletten (1995) we considered the detailed effects of an anisotropic permeability, whereas Ene (1991) presents a combined study of the effects of anisotropic permeability and diffusivity. However, the work contained in Ene (1991) considers only the leading order boundary layer flow, and also assumed that the principal axes of the permeability and diffusivity tensors were free to rotate in only one direction. Here we extend the pioneering work of Ene by performing a high-order boundary layer analysis, which takes into account the effect of the outer flow field, and by allowing the diffusivity tensor to have principal axes in any direction. We consider all inclinations of the heated surface which give boundary layer flow (except for small inclinations from the horizontal which results in a non-similar profile; see Rees and Riley (1985)) although we will concentrate on the generally inclined and vertical configurations rather than on the horizontal.

In this paper we shall determine the first three terms in the asymptotic expansion for the temperature and flowfield. This necessitates the use of the method of matched asymptotic expansions. In common with other applications of this method, the results we obtain for rates of heat transfer, for example, are, strictly speaking, only valid in the asymptotic limit of large values of x, the streamwise coordinate. However, in practice, it is often found that such asymptotic series provide accurate results even at fairly small, finite, values of x. The purpose of extending the work of Ene (1991) lies in reducing the values of x above which the boundary layer solution is an accurate representation of the solution of the full elliptic equations of motion. Again, in common with other studies of this kind, it is difficult to give an indication of the value of x above which our analysis is valid without performing the fully elliptic computations. Another restriction on the present analysis is given by considering the stability of the resulting flow. Although such a topic is the subject of the analysis of Storesletten and Rees (1997), the work described therein is applicable only to isotropic media. However, it is likely that similar qualitative results hold for anisotropic media, and therefore it is distinctly possible that the streamwise coordinate beyond which disturbances grow recedes to infinity as the upward-facing heated surface tends to the vertical, and, moreover, the boundary layer will be stable if the heated surface is downward-facing.

In Section 2 we derive the governing equations for anisotropic flow induced by a heated inclined surface. The leading-order boundary layer analysis is presented in Section 3 and this is extended to third order in Section 4 where we present a surprisingly simple expression for the rate of heat transfer from the heated surface. The results are discussed briefly in Section 5.

2. Mathematical Formulation

We consider the free convective boundary layer flow in a fluid-saturated porous medium with an anisotropic thermal diffusivity. The flow is induced by heating uniformly a semi-infinite surface embedded in the medium. The heated surface is maintained at a dimensional temperature T_w, whilst the ambient temperature is T_∞, where $T_w > T_\infty$. A Cartesian frame of reference is chosen where the x-axis is aligned with the heated surface, the y-axis is normal to the surface, and the z-axis

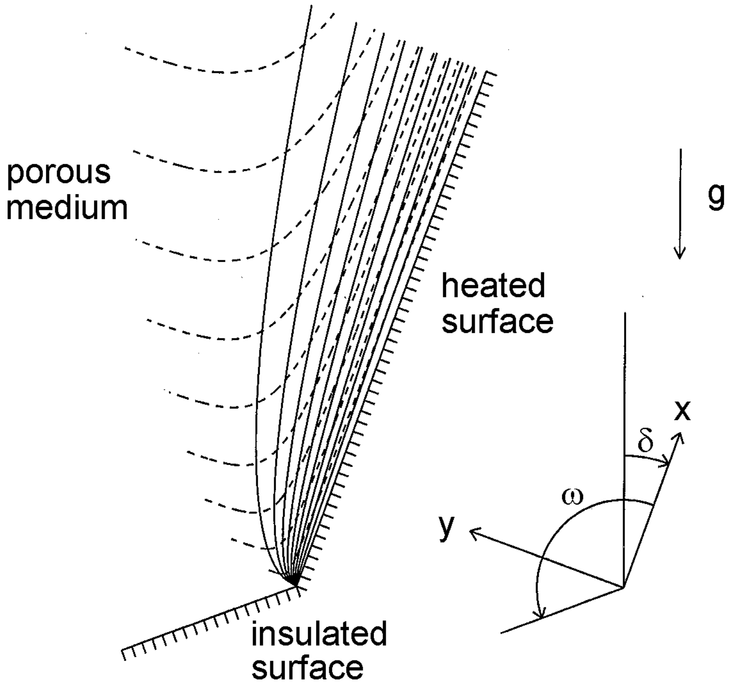

Figure 1. Definition sketch of the configuration and coordinate system. The z-coordinate is such that the axes form a right-handed coordinate system. Also shown are representative streamlines (dashed lines) and isotherms (solid line) for typical boundary layer flow.

is in the spanwise direction and is horizontal. The detailed configuration is shown in Figure 1. The diffusivity tensor, **D**, is given by

$$\mathbf{D} = D_1\hat{\mathbf{i}}\hat{\mathbf{i}} + D_2\hat{\mathbf{j}}\hat{\mathbf{j}} + D_3\hat{\mathbf{k}}\hat{\mathbf{k}}, \tag{1}$$

where the right-handed set of unit vectors, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are obtained by rotating the unit vectors in the x, y and z directions (respectively, **i**, **j** and **k**) by an angle α about the x-axis, followed by a rotation of angle β about the y-axis and an angle γ about the z-axis, in that order. In other words we have,

$$\begin{aligned}
\hat{\mathbf{i}} &= (\cos\beta\cos\gamma\,, \cos\beta\sin\gamma\,, -\sin\beta\,),\\
\hat{\mathbf{j}} &= (-\cos\alpha\sin\gamma\, + \sin\alpha\sin\beta\cos\gamma\,,\\
&\quad \cos\alpha\cos\gamma\, + \sin\alpha\sin\beta\sin\gamma\,, \sin\alpha\cos\beta\,),\\
\hat{\mathbf{k}} &= (\sin\alpha\sin\gamma\, + \cos\alpha\sin\beta\cos\gamma\,,\\
&\quad -\sin\alpha\cos\gamma\, + \cos\alpha\sin\beta\sin\gamma\,, \cos\alpha\cos\beta\,).
\end{aligned} \tag{2}$$

The symmetric diffusivity matrix can be written in the form,

$$\mathbf{D} = D_1 \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{pmatrix} = D_1 \mathbf{E}, \tag{3}$$

where D_1 is used as a reference diffusivity. Here the E_{ij} values are identical to the L_{ij} values in Rees and Storesletten (1995) when K_1, K_2 and K_3 are replaced by D_1, D_2 and D_3, and they are given in the Appendix.

Steady flow in a porous medium is governed by the equations

$$\operatorname{div} \mathbf{u} = 0, \tag{4a}$$

$$\mathbf{u} = -\frac{K}{\mu}(\nabla p + (T - T_\infty)\rho_0 b \mathbf{g}), \tag{4b}$$

$$\mathbf{u} \cdot \nabla T = \operatorname{div}(\mathbf{D} \cdot \nabla T), \tag{4c}$$

where Darcy's law and the Boussinesq approximation have been used and the density is assumed to be a linear function of the temperature, T, only. Moreover, μ is the fluid viscosity, K, the permeability, ρ_0, the density of the fluid at temperature T_∞, b, the coefficient of cubical expansion, $\mathbf{g}$, the gravity vector, $\mathbf{u} = (u, v, w)$, the velocity flux, and p, the dynamic pressure. We nondimensionalise by setting

$$(x, y, z) = \frac{\mu D_1}{\rho_0 g b K (T_w - T_\infty)}(x^*, y^*, z^*), \tag{5a}$$

$$\mathbf{u} = \frac{\rho_0 g b K (T_w - T_\infty)}{\mu}\mathbf{u}^*, \tag{5b}$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \quad \text{and} \quad p = \frac{\mu D_1}{K} p^* \tag{5c,d}$$

into (4). On omitting the asterisks we obtain

$$\operatorname{div} \mathbf{u} = 0, \tag{6a}$$

$$u = -p_x + \theta \cos \delta, \tag{6b}$$

$$v = -p_y + \theta \sin \delta, \tag{6c}$$

$$w = -p_z, \tag{6d}$$

$$u\theta_x + v\theta_y + w\theta_z = E_{11}\theta_{xx} + E_{22}\theta_{yy} + E_{33}\theta_{zz} + 2(E_{12}\theta_{xy} + E_{13}\theta_{xz} + E_{23}\theta_{yz}), \tag{6e}$$

where δ is the inclination of the surface from the vertical, and positive values of δ (subject to $\delta < \pi/2$) correspond to an upward facing surface. We note that there is no nondimensional parameter in this problem. This is a consequence of the fact that there is no natural length scale, but rather that the material parameters of the fluid and the medium define a length scale according to (5a).

In this study we consider the flow and temperature fields to be independent of z, even though the medium is anisotropic. An examination of (6e) suffices to show that z-independence is achieved when w is uniform, for although z-derivatives appear in the governing equations only a z-dependent spanwise velocity will cause a z-dependent solution. Hence, Equations (6) reduce to

$$\psi_{xx} + \psi_{yy} = \theta_y \cos\delta - \theta_x \sin\delta, \tag{7a}$$

$$E_{11}\theta_{xx} + 2E_{12}\theta_{xy} + E_{22}\theta_{yy} = \psi_y\theta_x - \psi_x\theta_y, \tag{7b}$$

$$w = \text{constant}, \tag{7c}$$

where the pressure p has been eliminated, and a streamfunction, ψ, is defined in the usual way by

$$u = \psi_y, \qquad v = -\psi_x. \tag{8}$$

Since w is a constant, according to (7c), there is a fluid drift in the z-direction, but this does not affect, nor is induced by the buoyancy-induced flow up the heated surface. This immediately tells us that the effects of an anisotropic thermal diffusivity are in general qualitatively different from the effects of an anisotropic permeability, since the latter induces a nonuniform spanwise fluid drift in general.

3. Boundary Layer Analysis

In this section we develop the leading order boundary layer theory for convection induced in a porous medium with an anisotropic diffusivity tensor. We follow the same procedure as that of Rees and Storesletten (1995).

On invoking the boundary layer approximation, which is equivalent to assuming that $x \gg y$, Equations (7a,b) reduce to

$$\psi_{yy} = \theta_y \cos\delta, \tag{9a}$$

$$E_{22}\theta_{yy} = (\psi_y\theta_x - \psi_x\theta_y). \tag{9b}$$

These equations are to be solved subject to the boundary conditions,

$$\psi = 0, \qquad \theta = 1 \quad \text{when } \; y = 0, \tag{10a}$$

$$\psi_y, \theta \to 0 \quad \text{as } \; y \to \infty. \tag{10b}$$

Equations (9) admit the similarity solution

$$\begin{aligned} \psi &= (E_{22}\cos\delta)^{1/2} x^{1/2} F((\cos\delta/E_{22})^{1/2}\eta), \\ \theta &= G((\cos\delta/E_{22})^{1/2}\eta), \end{aligned} \tag{11}$$

where

$$\eta = y/x^{1/2} \tag{12}$$

is the similarity variable and where F and G satisfy

$$F'' = G', \qquad G'' + FG'/2 = 0, \tag{13a,b}$$

subject to the boundary conditions

$$F(0) = 0, \qquad F'(\zeta) \to 0 \quad \text{as } \zeta \to \infty, \tag{14}$$

$$G(0) = 1, \qquad G(\zeta) \to 0 \quad \text{as } \zeta \to \infty. \tag{15}$$

In (13) to (15) primes denote derivatives with respect to $\zeta = (\cos\delta/E_{22})^{1/2}\,\eta$. The solution of (13) subject to (14) and (15) is well-known and first appeared in the context of convection in isotropic porous media ($E_{22} = 1$) in Cheng and Minkowycz (1977).

From Equations (11) to (15) it follows that the primary effect of an anisotropic diffusivity, at leading order, is to change the boundary layer thickness from that of an isotropic medium. The boundary layer is thicker when $E_{22} > 1$, has the same thickness when $E_{22} = 1$, and is thinner when $E_{22} < 1$. In the general case the quantity, E_{22}, is found to be

$$E_{22} = \frac{D_1(\cos\beta\sin\gamma)^2 + D_2(\sin\alpha\sin\beta\sin\gamma - \cos\alpha\cos\gamma)^2 +}{D_1}$$

$$\frac{+D_3(\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma)^2}{D_1}. \tag{16}$$

Clearly, E_{22} depends on the diffusivity ratios, D_2/D_1 and D_3/D_1, and on the three rotation angles, α, β and γ.

Let us consider three special cases, corresponding to the rotation of the principal axes about only one axis

(i) rotation about the x-axis,
(ii) about the y-axis, and
(iii) about the z-axis.

Case 1: Rotation about the x-axis.
Since both $\beta = 0$ and $\gamma = 0$, it follows that

$$E_{22} = \frac{D_2\cos^2\alpha + D_3\sin^2\alpha}{D_1}. \tag{17}$$

In this case E_{22} depends on both the rotation angle, α, and the two diffusivity ratios, D_2/D_1 and D_3/D_1. Thus the thickness of the boundary layer depends on the projections of the diffusivity components, D_2 and D_3 (in the $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ directions) on the y-axis, i.e., it depends on $D_2\cos\alpha$ and $-D_3\sin\alpha$, respectively.

Case 2: Rotation about the y-axis.
Since both $\alpha = 0$ and $\gamma = 0$, it follows that

$$E_{22} = \frac{D_2}{D_1}. \tag{18}$$

Clearly the thickness of the boundary layer is independent of the rotation angle and the third diffusivity component. The boundary layer is thicker than, as thick as, or thinner than that of an isotropic medium when $D_1 < D_2$, $D_1 = D_2$ or $D_1 > D_2$, respectively.

Case 3: Rotation about the z-axis.
Since both $\alpha = 0$ and $\beta = 0$, it follows that

$$E_{22} = \frac{D_1 \sin^2\gamma + D_2 \cos^2\gamma}{D_1}. \tag{19}$$

Consequently, the boundary layer is thicker than that of an isotropic medium when $E_{22} < 1$, which is satisfied if and only if $D_1 < D_2$ and $\gamma \neq \pi/2, 3\pi/2$. Conversely, the boundary layer is thinner when $D_1 > D_2$ and $\gamma \neq \pi/2, 3\pi/2$. It is of the same thickness and, hence, is indistinguishable from the isotropic case at leading order, when either $D_1 = D_2$ or $\gamma = \pi/2, 3\pi/2$.

As might be expected on physical grounds, the magnitude of D_3 does not affect the flow at leading order. More precisely, the thickness of the boundary layer depends on the projections of the diffusivity components, D_1 and D_2 (in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions) on the y-axis, i.e., it depends on $D_1 \sin\gamma$ and $D_2 \cos\gamma$, respectively.

4. Higher-Order Analysis

In this section we will describe as briefly as possible the higher-order boundary layer theory for convective flow in a porous medium with an anisotropic diffusivity tensor. The aim is to obtain a more account of the effects of anisotropy, and in particular the effects on the rate of heat transfer. We use the method of matched asymptotic expansions to determine an series solution for the flow within both the boundary layer and the outer region; in this regard we follow the procedure outlined in Riley and Rees (1985) and Storesletten and Rees (1997), and which is given in more detail in Chang and Cheng (1983), Cheng and Hsu (1984) and especially Daniels and Simpkins (1984).

In the boundary layer region the solution takes the following form as $x \to \infty$

$$\psi = x^{1/2} f_0(\eta) + f_1(\eta) + x^{-1/2} \ln x \bar{f}_2(\eta) + x^{-1/2} f_2(\eta) + \cdots, \tag{20a}$$

$$\theta = g_0(\eta) + x^{-1/2} g_1(\eta) + x^{-1} \ln x \bar{g}_2(\eta) + x^{-1} g_2(\eta) + \cdots\ . \tag{20b}$$

When these expressions are substituted into the basic Equations (7a) and (7b), and like powers of x are equated, then the following set of ordinary differential equations for the coefficient functions are obtained

$$f_0'' - g_0' \cos\delta = 0, \tag{21a}$$

$$E_{22} g_0'' + \tfrac{1}{2} f_0 g_0' = 0, \tag{21b}$$

$$f_1'' - g_1' \cos\delta = \tfrac{1}{2}\eta g_0' \sin\delta, \tag{21c}$$

$$E_{22} g_1'' + \tfrac{1}{2}(f_0 g_1' + f_0' g_1) = E_{12}(g_0' + \eta g_0''), \tag{21d}$$

$$\bar{f}_2'' - \bar{g}_2' \cos\delta = 0, \tag{21e}$$

$$E_{22}\bar{g}_2'' + \tfrac{1}{2}(f_0 \bar{g}_2' + 2 f_0' \bar{g}_2 - \bar{f}_2 g_0') = 0, \tag{21f}$$

$$f_2'' - g_2' \cos\delta = \tfrac{1}{2}(g_1 + \eta g_1') \sin\delta + \tfrac{1}{4}(f_0 - \eta f_0' - \eta^2 f_0''), \tag{21g}$$

$$\begin{aligned} &E_{22} g_2'' + \tfrac{1}{2}(f_0 g_2' + 2 f_0' g_2 - f_2 g_0') \\ &\quad = 2E_{12}(g_1' + \tfrac{1}{2}\eta g_1'') - \tfrac{1}{4}E_{11}(3\eta g_0' + \eta^2 g_0'') - \tfrac{1}{2} f_1' g_1 + f_0' \bar{g}_2 - \bar{f}_2 g_0'. \end{aligned} \tag{21h}$$

At $\eta = 0$ these functions satisfy the boundary conditions,

$$f_0 = f_1 = \bar{f}_2 = f_2 = 0, \tag{22a}$$

$$g_0 = 1, \qquad g_1 = \bar{g}_2 = g_2 = 0, \tag{22b}$$

whilst the appropriate conditions which match asymptotically onto the outer flow are that

$$f_0', \quad \bar{f}_2, \quad g_0, \quad g_1, \quad \bar{g}_2, \quad g_2 \to 0, \tag{22c}$$

$$f_1' \to -\tfrac{1}{2} a_0 \cot(\omega/2) \quad \text{and} \quad f_2' \to \tfrac{1}{4}\eta a_0 - a_1/\omega \tag{22d}$$

as $\eta \to \infty$, where the constants a_0 and a_1 are given by the limiting forms,

$$f_0 \to a_0, \quad \text{and} \quad f_1 - \eta f_1' \to a_1 \quad \text{as } \eta \to \infty. \tag{22e}$$

In (22d) ω is the wedge angle between the plane surfaces bounding the porous medium, as sketched in Figure 1. The matching conditions, (22c) and (22d), were obtained by considering the flow in the region exterior to the boundary layer where we have set

$$\psi \sim r^{1/2} F_0(\phi) + \ln r F_1(\phi) + F_2(\phi), \qquad \theta \sim O(\exp). \tag{23}$$

The details of such an analysis are now well known and are similar to those presented in the last set of references quoted above.

Equations (21e) and (21f), which correspond to the logarithmic terms in (20), are homogeneous and admit an eigensolution which can be written in the form,

$$\bar{f}_2 = \lambda(\eta f_0' - f_0), \qquad \bar{g}_2 = \lambda \eta g_0', \tag{24}$$

where λ is an amplitude which can be obtained by insisting that Equations (21g) and (21h) have a solution; this analytical form of the eigensolution was first presented for an isotropic medium in Daniels and Simpkins (1984). Thus λ may be found numerically by imposing a further boundary condition on the solution for g_2 – in our numerical computations we selected the value $g_2'(0) = 0$, but any other choice would yield the same eigenvalue. The computation of λ would seem to be quite a lengthy process since it will, in general, be a function of the five parameters, E_{11}, E_{12}, E_{22}, δ and ω. However, it is possible to reduce very substantially the numerical effort by a suitable rescaling of Equations (21). If we define a scaled similarity variable ζ according to

$$\eta = (E_{22}/\cos\delta)^{1/2}\zeta, \tag{25}$$

and introduce the substitutions,

$$f_0(\eta) = (E_{22}\cos\delta)^{1/2}F_0(\zeta), \qquad g_0(\eta) = G_0(\zeta), \tag{26a,b}$$

$$f_1(\eta) = E_{12}F_1(\zeta) + E_{22}\tan\delta\,\hat{F}_1(\zeta) - \tfrac{1}{2}[E_{22}A_0\cot(\omega/2)]\zeta, \tag{26c}$$

$$g_1(\eta) = E_{12}(E_{22}\cos\delta)^{-1/2}G_1(\zeta), \tag{26d}$$

$$\begin{aligned} f_2(\eta) = (E_{22}\cos\delta)^{1/2}\Bigg[&\left(\frac{E_{12}\sin\delta}{\cos^2\delta}\right)F_{21}(\zeta) + \left(\frac{E_{22}}{\cos\delta}\right)F_{22}(\zeta) + \\ &+ \left(\frac{E_{12}^2}{E_{22}\cos\delta}\right)F_{23}(\zeta) + \left(\frac{E_{11}}{\cos\delta}\right)F_{24}(\zeta) + \\ &+ \left(\frac{E_{12}A_0\cot(\omega/2)}{\cos\delta}\right)F_{25}(\zeta) + \\ &+ \left(\frac{E_{12}A_1}{\omega\cos\delta}\right)(F_{26}(\zeta) - \zeta) + \\ &+ \left(\frac{E_{22}\hat{A}_1\sin\delta}{\omega\cos^2\delta}\right)(F_{27}(\zeta) - \zeta)\Bigg], \end{aligned} \tag{26e}$$

$$\begin{aligned} g_2(\eta) = &\left(\frac{E_{12}\sin\delta}{\cos^2\delta}\right)G_{21}(\zeta) + \left(\frac{E_{22}}{\cos\delta}\right)G_{22}(\zeta) + \\ &+ \left(\frac{E_{12}^2}{E_{22}\cos\delta}\right)G_{23}(\zeta) + \left(\frac{E_{11}}{\cos\delta}\right)G_{24}(\zeta) + \\ &+ \left(\frac{E_{12}A_0\cot(\omega/2)}{\cos\delta}\right)G_{25}(\zeta) + \left(\frac{E_{12}A_1}{\omega\cos\delta}\right)G_{26}(\zeta) + \\ &+ \left(\frac{E_{22}\hat{A}_1\sin\delta}{\omega\cos^2\delta}\right)G_{27}(\zeta), \end{aligned} \tag{26f}$$

and

$$\lambda = \left(\frac{E_{12}\sin\delta}{\cos^2\delta}\right)\lambda_1 + \left(\frac{E_{22}}{\cos\delta}\right)\lambda_2 + \left(\frac{E_{12}^2}{E_{22}\cos\delta}\right)\lambda_3 + {}$$
$$+ \left(\frac{E_{11}}{\cos\delta}\right)\lambda_4 + \left(\frac{E_{12}A_0\cot(\omega/2)}{\cos\delta}\right)\lambda_5 + {}$$
$$+ \left(\frac{E_{12}A_1}{\omega\cos\delta}\right)\lambda_6 + \left(\frac{E_{22}\hat{A}_1\sin\delta}{\omega\cos^2\delta}\right)\lambda_7, \tag{26g}$$

then we obtain the equations

$$F_0'' - G_0' = 0, \qquad G_0'' + \tfrac{1}{2}F_0G_0' = 0, \tag{27a,b}$$

$$F_1'' - G_1' = 0, \qquad G_1'' + \tfrac{1}{2}(F_0G_1' + F_0'G_1) = G_0' + \zeta G_0'', \tag{27c,d}$$

$$\hat{F}_1'' = \tfrac{1}{2}\zeta G_0', \tag{27e}$$

$$\mathcal{L}_1(F_{21}, G_{21}) = \tfrac{1}{2}(G_1 + \zeta G_1'), \tag{27f}$$

$$\mathcal{L}_2(F_{21}, G_{21}) = -\tfrac{1}{2}\hat{F}_1'G_1 + \lambda_1F_0G_0', \tag{27g}$$

$$\mathcal{L}_1(F_{22}, G_{22}) = \tfrac{1}{4}(F_0 - \zeta F_0' - \zeta^2F_0''), \tag{27h}$$

$$\mathcal{L}_2(F_{22}, G_{22}) = \lambda_2F_0G_0', \tag{27i}$$

$$\mathcal{L}_1(F_{23}, G_{23}) = 0, \tag{27j}$$

$$\mathcal{L}_2(F_{23}, G_{23}) = 2G_1' + \zeta G_1'' - \tfrac{1}{2}F_1'G_1 + \lambda_3F_0G_0', \tag{27k}$$

$$\mathcal{L}_1(F_{24}, G_{24}) = 0, \tag{27l}$$

$$\mathcal{L}_2(F_{24}, G_{24}) = -\tfrac{1}{4}(3\zeta G_0' + \zeta^2G_0'') + \lambda_4F_0G_0', \tag{27m}$$

$$\mathcal{L}_1(F_{25}, G_{25}) = 0, \qquad \mathcal{L}_2(F_{25}, G_{25}) = \tfrac{1}{4}G_1 + \lambda_5F_0G_0', \tag{27n,o}$$

$$\mathcal{L}_1(F_{26}, G_{26}) = 0, \qquad \mathcal{L}_2(F_{26}, G_{26}) = -\tfrac{1}{2}\zeta G_0' + \lambda_6F_0G_0', \tag{27p,q}$$

$$\mathcal{L}_1(F_{27}, G_{27}) = 0, \qquad \mathcal{L}_2(F_{27}, G_{27}) = -\tfrac{1}{2}\zeta G_0' + \lambda_7F_0G_0', \tag{27r,s}$$

where the differential operators, $\mathcal{L}_1$ and $\mathcal{L}_2$, are defined as

$$\begin{aligned}\mathcal{L}_1(\mathcal{F}, \mathcal{G}) &= \mathcal{F}'' - \mathcal{G}', \\ \mathcal{L}_2(\mathcal{F}, \mathcal{G}) &= \mathcal{G}'' + \tfrac{1}{2}F_0\mathcal{G}' + F_0'\mathcal{G} - \tfrac{1}{2}G_0'\mathcal{F}.\end{aligned} \tag{28}$$

The boundary conditions are that

$$\begin{aligned} &F_0 = 0, \qquad G_0 = 1, \qquad F_1 = 0, \qquad G_1 = 0, \\ &\hat{F}_1 = 0, \qquad F_{2n} = G_{2n} = 0 \quad (n = 1, 5), \end{aligned} \tag{29}$$

at $\zeta = 0$, and, as $\zeta \to \infty$, all the following functions tend to zero: F_0', F_1', $\hat{F}_1'$, F_{21}', $(F_{22}' - \frac{1}{4}A_0\zeta)$, F_{23}', F_{24}', F_{25}', F_{26}', F_{27}', G_0, G_1, G_{21}, G_{22}, G_{23}, G_{24}, G_{25}, G_{26} and G_{27}. In the above, the values, A_0, A_1 and $\hat{A}_1$ which appear are given by

$$F_0 \to A_0, \qquad F_1 \to A_1, \qquad \hat{F}_1 \to \hat{A}_1 \quad \text{as } \zeta \to \infty, \tag{30}$$

where we find numerically that $A_0 = 1.616125$ and $A_1 = -\hat{A}_1 = -2.28604$.

The numerical solution of the system of equations (27) was undertaken using a fourth order Runge–Kutta code incorporating the shooting method. The value of $\zeta_{\max} = 40$ together with a steplength of 0.05 were sufficient to ensure accuracy to more than five significant figures. However, the results we obtained, notably the value for λ_5 in comparison with that of A_0 and of λ_6 and λ_7, as shall be seen below, demonstrated that further analytical progress could be made.

Firstly we note that Equations (27c) to (27e) have the solutions,

$$F_1(\zeta) = \tfrac{1}{2}\zeta^2 F_0' - \zeta F_0 + \int_0^\zeta F_0(\xi)\,\mathrm{d}\xi, \qquad G_1(\zeta) = \tfrac{1}{2}\zeta^2 F_0'', \tag{31a,b}$$

and

$$\hat{F}_1 = \tfrac{1}{2}\zeta(F_0 + A_0) - \int_0^\zeta F_0(\xi)\,\mathrm{d}\xi. \tag{32}$$

Secondly, the evaluation of the values of λ_1 to λ_7 in integral form requires the use of a solvability condition and the derivation of an adjoint system of equations which corresponds to the differential operators given in (28). It is easily shown that all seven sets of equations given in (27f) to (27s) may be written in the form

$$\bar{F}''' + (\tfrac{1}{2}F_0)\bar{F}'' + (F_0')\bar{F} - (\tfrac{1}{2}F_0'')\bar{F} = \mathcal{R}, \tag{33}$$

where $\bar{F}(0) = \bar{F}'(0) = 0$ and $\bar{F}'(\zeta) \to 0$ as $\zeta \to \infty$. The adjoint system is obtained by multiplying (33) by $F_a(\zeta)$, the adjoint function to be found, and integrating between 0 and ∞. On integrating by parts we obtain

$$\int_0^\infty \bar{F}\left[-F_a''' + (\tfrac{1}{2}F_0)F_a'' - (F_0'')F_a\right]\mathrm{d}\zeta = \int_0^\infty \mathcal{R}F_a\,\mathrm{d}\zeta, \tag{34a}$$

and therefore the adjoint equation is obtained by setting the term in square brackets to zero

$$F_a''' - (\tfrac{1}{2}F_0)F_a'' + (F_0'')F_a = 0. \tag{34b}$$

It is easily verified that the Equation (34b) admits the solution $F_a = F_0$ and therefore the required solvability condition is simply that

$$\int_0^\infty \mathcal{R} F_0 \, \mathrm{d}\zeta = 0. \tag{35}$$

The application of this condition to Equations (27f) to (27o) yields the following results

$$\lambda_1 = 0.65297 = \tfrac{1}{4} A_0^2, \qquad \lambda_4 = -\lambda_2 = 0.32648 = \tfrac{1}{8} A_0^2, \tag{36a}$$

$$\lambda_3 = 0, \qquad \lambda_5 = -0.80806 = -\tfrac{1}{2} A_0 \tag{36b}$$

and

$$\lambda_6 = \lambda_7 = 1. \tag{36c}$$

The rate of heat transfer from the heated surface may now be evaluated using all the above information. Thus we find that

$$\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} = \left(\frac{\cos\delta}{E_{22}}\right)^{1/2} G_0'(0)[1 + \lambda x^{-1} \ln x + \mathrm{O}(x^{-1})], \tag{37}$$

where λ is given by

$$\begin{aligned} \lambda \; = \; & \frac{A_0^2}{\cos\delta}[\tfrac{1}{4} E_{12} \tan\delta + \tfrac{1}{8}(E_{11} - E_{22}) - \tfrac{1}{2} E_{12} \cot(\omega/2)] + \\ & + \frac{E_{12} A_1}{\omega \cos\delta} + \frac{E_{22} \hat{A}_1 \sin\delta}{\omega \cos^2\delta} \end{aligned} \tag{38}$$

and where $G_0'(0) = -0.44375$.

Given that λ_3 is zero, it is readily seen that the value of λ is a linear function of E_{11}, E_{12} and E_{22}, although its dependence on δ and ω is a little more complicated. In general, though, λ is a complicated function of seven variables: the three rotation angles, the two diffusivity ratios, the wedge angle and the slope of the heated surface. This result, (38), agrees with those numerical values presented in Riley and Rees (1985) when the medium is isotropic, although it must be noted that the definition of λ used in that paper differs slightly from that defined here.

5. Conclusion

In this paper we have considered the effects of an anisotropic thermal diffusivity on convection induced by a generally inclined, uniformly heated surface embedded in a fluid-saturated porous medium. In this regard the present study is complementary to that of Rees and Storesletten (1995) who considered in detail the effect of an anisotropic permeability. The analysis given above shows that the presence of anisotropy serves to change the thickness of the thermal boundary layer, as does the presence of an anisotropic permeability (see Rees and Storesletten (1995)). However, unlike the case of an anisotropic permeability, the presence of an anisotropic thermal diffusivity does not induce a spanwise fluid motion, though a uniform drift caused by a uniform external spanwise pressure gradient may be allowed to be present without modifying the boundary layer flow. The precise flow field, and the rate of heat transfer from the surface into the medium depend on the diffusivity ratios, all three rotation angles, the angle of inclination of the heated surface and the wedge-angle. We have demonstrated that the rate of heat transfer, correct to the appearance of the first logarithmic term in the boundary layer expansion, may be written down analytically subject to the computation of seven constants which have been presented. Thus, any combination of parameters may be considered easily without having to compute the solution of a large system of ordinary differential equations; we note that a similar conclusion cannot be given for a medium with an anisotropic permeability. As is usual for this type of analysis, further terms in the expansion for the rate of heat transfer cannot be obtained since (f_2, g_2) contains an abitrary multiple of the eigensolution, $(\bar{f}_2, \bar{g}_2)$.

Appendix

In this appendix we present the definitions of the entries of the diffusivity tensor given in Equation (3).

$$
\begin{aligned}
D_1E_{11} &= D_1\cos^2\beta\cos^2\gamma + D_2(\sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma)^2 + \\
&\quad + D_3(\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma)^2, \\
D_1E_{22} &= D_1\cos^2\beta\sin^2\gamma + D_2(\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma)^2 + \\
&\quad + D_3(\cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma)^2, \\
D_1E_{33} &= D_1\sin^2\beta + D_2\sin^2\alpha\cos^2\beta + D_3\cos^2\alpha\cos^2\beta\,.
\end{aligned}
$$

$$
\begin{aligned}
D_1E_{12} &= D_1\cos^2\beta\sin\gamma\cos\gamma + D_2(\sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma)\times \\
&\quad \times(\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma) + \\
&\quad + D_3(\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma)\times \\
&\quad \times(\cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma), \\
D_1E_{13} &= -D_1\sin\beta\cos\beta\cos\gamma + \\
&\quad + D_2(\sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma)\sin\alpha\cos\beta + \\
&\quad + D_3(\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma)\cos\alpha\cos\beta\,, \\
D_1E_{23} &= -D_1\sin\beta\cos\beta\sin\gamma +
\end{aligned}
$$

$$+D_2(\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma)\sin\alpha\cos\beta +$$
$$+D_3(\cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma)\cos\alpha\cos\beta\,.$$

Acknowledgement

The first author gratefully acknowledges financial support from the Norwegian Research Council during his sabbatical leave at the University of Bath, Autumn 1994.

References

Chang, I. D. and Cheng, P.: 1983, Matched asymptotic expansions for free convection about an impermeable horizontal surface in a porous medium, *Int. J. Heat Mass Transf.* **26**, 163–174.

Cheng, P. and Chang, I. D.: 1976, Buoyancy induced flows in a porous medium adjacent to impermeable horizontal boundaries, *Int. J. Heat Mass Transf.* **19**, 1267–1272.

Cheng, P. and Hsu, C. T.: 1984, Higher order approximations for Darcian free convection about a semi-infinite vertical flat plate, *Trans. ASME J. Heat Transfer* **106**, 143–151.

Cheng, P. and Minkowycz, W. J.: 1977, Free convection about a vertical flat plate imbedded in a porous medium with application to heat transfer from a dike, *J. Geophys. Res.* **82**, 2040–2044.

Daniels, P. G. and Simpkins, P. G.: 1984, The flow induced by a heated vertical wall in a porous medium, *Q. J. Mech. Appl. Math.* **37**, 339–354.

Degan, G., Vasseur, P. and Bilgen, E.: 1995, Convective heat transfer in a vertical anisotropic porous layer, *Int. J. Heat Mass Transfer* **38**, 1975–1987.

Ene, H. I.: 1991, Effects of anisotropy on the free convection from a vertical plate embedded in a porous medium, *Transport in Porous Media* **6**, 183–194.

Kvernvold, O. and Tyvand, P. A.: 1979, Nonlinear thermal convection in anisotropic porous media, *J. Fluid Mech.* **90**, 609–624.

Ni, J. and Beckermann, C.: 1991, Natural convection in a vertical enclosure filled with anisotropic porous media, *Trans. ASME J. Heat Transfer* **113**, 1033–1037.

Rees, D. A. S. and Riley, D. S.: 1985, Free convection above a near horizontal semi-infinite heated surface embedded in a saturated porous medium, *Int. J. Heat Mass Transfer* **28**, 183–190.

Rees, D. A. S. and Storesletten, L.: 1995, The effect of anisotropic permeability on free convective boundary layer flow in porous media, *Transport in Porous Media* **19**, 79–92.

Riley, D. S. and Rees, D. A. S.: 1985, Non-Darcy natural convection from arbitrarily inclined heated surfaces in saturated porous media, *Q. J. Mech. Appl. Math.* **38**, 277–295.

Storesletten, L.: 1993, Natural convection in a horizontal porous layer with anisotropic thermal diffusivity, *Transport in Porous Media* **12**, 19–29.

Storesletten, L. and Rees, D. A. S.: 1997, The influence of higher-order effects on the linear instability of thermal boundary layer flow in porous media. Submitted to *Int. J. Heat Mass Transfer*

Tyvand, P. A. and Storesletten, L.: 1991, Onset of convection in an anisotropic porous medium with oblique principal axes, *J. Fluid Mech.* **226**, 371–382.