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THE EFFECT OF LONGITUDINAL SURFACE WAVES ON FREE CONVECTION FROM VERTICAL SURFACES IN POROUS MEDIA

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ABSTRACT

In this paper we consider the effect of longitudinal surface waves on the thermal boundary layer flow induced by a vertically aligned heated surface embedded in a porous medium. The full governing equations are considered and the boundary layer equations are derived in a systematic way. It is found that, for a wide range of values of x , the distance from the leading edge, the boundary layer equations for the three-dimensional flowfield are satisfied by a two-dimensional similarity solution.

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Introduction

In this note we consider free convection induced by vertically orientated surfaces which are not planar but exhibit steady longitudinal surface waves, as depicted in Fig. 1. The wavelength of the waves is comparable with their amplitude and the Darcy-Rayleigh number is assumed to be large in order to consider the effect of surface waves on boundary layer flow. This work extends a recent series of papers by Rees and Pop [1–3] which are concerned with the effects of stationary transverse surface waves on vertical and horizontal surfaces. To our knowledge this is the first study of the effect of longitudinal waves for which the resulting flow-field is three-dimensional.

We consider here two different cases, namely, a prescribed power-law variation of the surface temperature, and a prescribed power-law variation of the surface heat flux. For both cases it is shown that the fully three-dimensional equations of motion and energy conservation which include

the effects of transverse surface curvature and which are subject to the usual boundary layer approximation, remain satisfied by similarity solutions. Conditions are given for the validity of these solutions.

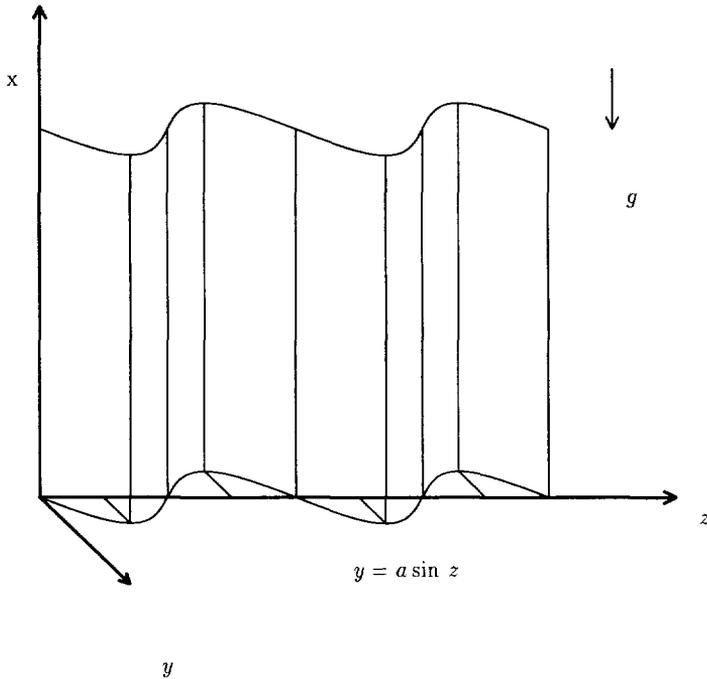


FIG 1

Physical Model and Coordinate System depicting Longitudinal Surface Waves

Analysis

The nondimensional equations of motion governing steady Darcy–Boussinesq free convection flow for this problem are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$u = -\frac{\partial p}{\partial x} + Ra \theta \tag{2}$$

$$v = -\frac{\partial p}{\partial y}, \quad w = -\frac{\partial p}{\partial z} \tag{3,4}$$

$$\nabla^2 \theta = u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \tag{5}$$

where (x, y, z) are Cartesian coordinates, (u, v, w) are the respective velocity components, p is the pressure, Ra is the Rayleigh number, and ∇^2 is the three-dimensional Laplacian operator.

We shall solve these equations for the case of (i) a prescribed surface temperature, and (ii) a prescribed surface heat flux. Thus the boundary conditions for Eqs. (1) to (5) are

$$\left. \begin{aligned} \underline{n} \cdot \underline{u} &= 0, & (i) \quad \theta &= x^\lambda \\ & & (ii) \quad \underline{n} \cdot \nabla \theta &= -x^\lambda \end{aligned} \right\} \text{ on } y = a \sin z \tag{6}$$

$$\underline{u}, \theta \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty$$

where \underline{n} is the unit vector normal to the heated surface.

The three velocity components can be eliminated to obtain the pressure-temperature formulation,

$$\nabla^2 p = Ra \frac{\partial \theta}{\partial x} \quad \nabla^2 \theta = Ra \theta \frac{\partial \theta}{\partial x} - \nabla p \cdot \nabla \theta. \tag{7, 8}$$

Then, in order to 'straighten out' the wavy boundaries, we will introduce a coordinate transformation defined by

$$\hat{x} = x, \quad \hat{y} = y - a \sin z, \quad \hat{z} = z. \tag{9}$$

Hence Eqs. (7) and (8) become

$$\nabla_1^2 p = Ra \frac{\partial \theta}{\partial \hat{x}} \tag{10}$$

$$\begin{aligned} \nabla_1^2 \theta &= Ra \theta \frac{\partial \theta}{\partial \hat{x}} - \frac{\partial p}{\partial \hat{x}} \frac{\partial \theta}{\partial \hat{x}} - (1 + a^2 \cos^2 \hat{z}) \frac{\partial p}{\partial \hat{y}} \frac{\partial \theta}{\partial \hat{y}} \\ &\quad - \frac{\partial p}{\partial \hat{z}} \frac{\partial \theta}{\partial \hat{z}} + a \cos \hat{z} \left(\frac{\partial p}{\partial \hat{z}} \frac{\partial \theta}{\partial \hat{y}} + \frac{\partial p}{\partial \hat{y}} \frac{\partial \theta}{\partial \hat{z}} \right) \end{aligned} \tag{11}$$

where

$$\begin{aligned} \nabla_1^2 &= \frac{\partial^2}{\partial \hat{x}^2} + (1 + a^2 \cos^2 \hat{z}) \frac{\partial^2}{\partial \hat{y}^2} + \frac{\partial^2}{\partial \hat{z}^2} \\ &\quad - 2a \cos \hat{z} \frac{\partial^2}{\partial \hat{y} \partial \hat{z}} + a \sin \hat{z} \frac{\partial}{\partial \hat{y}}. \end{aligned} \tag{12}$$

The boundary conditions (6) also become

$$(1 + a^2 \cos^2 \hat{z}) \frac{\partial p}{\partial \hat{y}} = a \cos \hat{z} \frac{\partial p}{\partial \hat{z}}$$

$$(i) \quad \theta = \hat{x}^\lambda \tag{13a}$$

$$(ii) \quad (1 + a^2 \cos^2 \hat{z}) \frac{\partial \theta}{\partial \hat{y}} = a \cos \hat{z} \frac{\partial \theta}{\partial \hat{z}} - x^\lambda (1 + a^2 \cos^2 \hat{z})$$

on $\hat{y} = 0$, and

$$\frac{\partial^2 p}{\partial \hat{y}^2}, \theta \longrightarrow 0 \quad \text{as} \quad \hat{y} \longrightarrow \infty. \tag{13b}$$

It is now assumed that the boundary layer approximation is valid, or, more precisely, that the Rayleigh number is asymptotically large. Thus, convection takes place primarily within a thin

layer adjacent to the heated surface. Since Ra is based on the wavelength, l , of the surface waves, this means that the boundary layer thickness is asymptotically smaller than the surface wavelength at $O(1)$ values of \hat{x} when $Ra \rightarrow \infty$.

We will now treat separately the two cases of prescribed surface temperature and prescribed heat flux at the surface.

(i) Prescribed surface temperature.

For this case we introduce the boundary layer variables, (ξ, η) , where

$$\xi = \hat{x}, \quad \eta = \hat{y} Ra^{1/2} / \hat{x}^{(1-\lambda)/2}. \tag{14}$$

Hence, the governing Eqs. (10) and (11) can be written as

$$\mathcal{L}p = Ra \left(\frac{\partial \theta}{\partial \xi} - \frac{(1-\lambda)\eta}{2} \frac{\partial \theta}{\xi \partial \eta} \right) \tag{15}$$

$$\begin{aligned} \mathcal{L}\theta = & Ra\theta \left(\frac{\partial \theta}{\partial \xi} - \frac{1-\lambda}{2} \frac{\eta}{\xi} \frac{\partial \theta}{\partial \eta} \right) - \left(\frac{\partial p}{\partial \xi} - \frac{(1-\lambda)\eta}{2} \frac{\partial p}{\xi \partial \eta} \right) \left(\frac{\partial \theta}{\partial \xi} - \frac{(1-\lambda)\eta}{2} \frac{\partial \theta}{\xi \partial \eta} \right) \\ & - Ra \xi^{\lambda-1} (1 + a^2 \cos^2 \hat{z}) \frac{\partial p}{\partial \eta} \frac{\partial \theta}{\partial \eta} - \frac{\partial p}{\partial \hat{z}} \frac{\partial \theta}{\partial \hat{z}} + a \cos \hat{z} Ra^{1/2} \xi^{(\lambda-1)/2} \left(\frac{\partial p}{\partial \hat{z}} \frac{\partial \theta}{\partial \eta} + \frac{\partial p}{\partial \eta} \frac{\partial \theta}{\partial \hat{z}} \right) \end{aligned} \tag{16}$$

subject to the boundary conditions

$$(1 + a^2 \cos^2 \hat{z}) Ra^{1/2} \xi^{(\lambda-1)/2} \frac{\partial p}{\partial \eta} = a \cos \hat{z} \frac{\partial p}{\partial \hat{z}}, \quad \theta = \xi^\lambda \quad \text{on} \quad \eta = 0 \tag{17a}$$

$$\frac{\partial^2 p}{\partial \eta^2}, \theta \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \tag{17b}$$

The differential operator, \mathcal{L} , is defined as

$$\begin{aligned} \mathcal{L} = & \frac{\partial^2}{\partial \xi^2} + \frac{(1-\lambda)^2 \eta^2}{4} \frac{\partial^2}{\xi^2 \partial \eta^2} - (1-\lambda) \frac{\eta}{\xi} \frac{\partial^2}{\partial \xi \partial \eta} \\ & + \frac{(3-\lambda)(1-\lambda)}{4} \frac{\eta}{\xi^2} \frac{\partial}{\partial \eta} + (1 + a^2 \cos^2 \hat{z}) \frac{Ra}{\xi^{1-\lambda}} \frac{\partial^2}{\partial \eta^2} \\ & + \frac{\partial^2}{\partial \hat{z}^2} - 2a \cos \hat{z} \frac{Ra^{1/2}}{\xi^{(1-\lambda)/2}} \frac{\partial^2}{\partial \eta \partial \hat{z}} + a \sin \hat{z} \frac{Ra^{1/2}}{\xi^{(1-\lambda)/2}} \frac{\partial}{\partial \eta}. \end{aligned} \tag{18}$$

We set

$$\theta = \xi^\lambda g(\eta) \tag{19}$$

and when the Rayleigh number is large, the leading order terms in Eqs. (15) and (16) are

$$(1 + a^2 \cos^2 \hat{z}) \frac{\partial^2 p}{\partial \eta^2} = \lambda g - \frac{(1-\lambda)}{2} \eta \frac{\partial g}{\partial \eta} \tag{20}$$

$$(1 + a^2 \cos^2 \hat{z}) \frac{\partial^2 \theta}{\partial \eta^2} = \lambda g^2 - \frac{(1-\lambda)}{2} \eta g \frac{\partial g}{\partial \eta} - (1 + a^2 \cos^2 \hat{z}) \frac{\partial p}{\partial \eta} \frac{\partial g}{\partial \eta}. \tag{21}$$

A further transformation,

$$\eta = \bar{\eta}(1 + a^2 \cos^2 \hat{z})^{1/2} \tag{22}$$

reduces Eqs. (20) and (21) to

$$\frac{\partial^2 p}{\partial \bar{\eta}^2} = \lambda g - \frac{(1 - \lambda)}{2} \bar{\eta} \frac{\partial g}{\partial \bar{\eta}} \tag{23}$$

$$\frac{\partial^2 \theta}{\partial \bar{\eta}^2} = \lambda g^2 - \frac{(1 - \lambda)}{2} \bar{\eta} g \frac{\partial g}{\partial \bar{\eta}} - \frac{\partial p}{\partial \bar{\eta}} \frac{\partial g}{\partial \bar{\eta}}. \tag{24}$$

The solution of Eqs. (23) and (24) is

$$p = \int_0^{\bar{\eta}} f(t) dt + \frac{(\lambda - 1)}{2} \bar{\eta} f(\bar{\eta}), \quad g = f'(\bar{\eta}) \tag{25a, b}$$

where f is given by the equation

$$f''' + \frac{1 + \lambda}{2} f f'' - \lambda f' f' = 0 \tag{26a}$$

subject to

$$f(0) = 0, \quad f'(0) = 1, \quad \text{and} \quad f'(\bar{\eta}) \rightarrow 0 \quad \text{as} \quad \bar{\eta} \rightarrow \infty; \tag{26b}$$

here primes denote differentiation with respect to $\bar{\eta}$.

Equation (26) is precisely that obtained by Cheng and Minkowycz [4] in their study of thermal boundary layer flow induced by a heated vertical surface embedded in a porous medium.

(ii) Prescribed surface heat flux.

The appropriate boundary layer variables are now

$$\xi = \hat{x}, \quad \eta = \frac{Ra^{1/3}}{\xi^{(1-\lambda)/3}} \hat{y}. \tag{27}$$

instead of (14), and we take

$$\theta = Ra^{-1/3} \xi^{(1+2\lambda)/3} g(\eta) \tag{28}$$

On substituting (28) into Eqs. (10) and (11) and taking the leading order terms when the Rayleigh number is large, we obtain

$$(1 + a^2 \cos^2 \hat{z}) \frac{\partial^2 p}{\partial \eta^2} = \frac{(1 + 2\lambda)}{3} g - \frac{(1 - \lambda)}{3} \eta \frac{\partial g}{\partial \eta} \tag{29}$$

$$(1 + a^2 \cos^2 \hat{z}) \frac{\partial^2 \theta}{\partial \eta^2} = \left(\frac{(1 + 2\lambda)}{3} g - \frac{(1 - \lambda)}{3} \eta \frac{\partial g}{\partial \eta} \right) g - (1 + a^2 \cos^2 \hat{z}) \frac{\partial p}{\partial \eta} \frac{\partial g}{\partial \eta}. \tag{30}$$

and the boundary conditions (13) become

$$\frac{\partial p}{\partial \eta} = 0, \quad (1 + a^2 \cos^2 \hat{z}) \frac{\partial g}{\partial \eta} = -1 \quad \text{at} \quad \eta = 0 \tag{31a}$$

$$\frac{\partial^2 p}{\partial \eta^2}, \quad \theta \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \tag{31b}$$

Equations (29)-(31) may be rendered independent of \hat{z} by means of the transformation given by (22). Hence, we have

$$\frac{\partial^2 p}{\partial \bar{\eta}^2} = \frac{(1 + 2\lambda)}{3}g - \frac{(1 - \lambda)}{3}\bar{\eta} \frac{\partial g}{\partial \bar{\eta}} \tag{32}$$

$$\frac{\partial^2 \theta}{\partial \bar{\eta}^2} = \left(\frac{(1 + 2\lambda)}{3}g - \frac{(1 - \lambda)}{3}\bar{\eta} \frac{\partial g}{\partial \bar{\eta}} \right) g - \frac{\partial p}{\partial \bar{\eta}} \frac{\partial g}{\partial \bar{\eta}} \tag{33}$$

subject to the boundary conditions

$$\frac{\partial p}{\partial \bar{\eta}} = 0, \quad \frac{\partial g}{\partial \bar{\eta}} = -1 \quad \text{at} \quad \bar{\eta} = 0 \quad \text{and} \quad \frac{\partial^2 p}{\partial \bar{\eta}^2}, \theta \rightarrow 0 \quad \text{as} \quad \bar{\eta} \rightarrow \infty. \tag{34}$$

The solution of Eqs. (32) and (33) is

$$p = \int_0^{\bar{\eta}} f(t) dt + \frac{\lambda - 1}{3}\bar{\eta}f(\bar{\eta}), \quad g = f'(\bar{\eta}) \tag{35}$$

where f is now given by the equation

$$f''' + \frac{2 + \lambda}{3}ff'' - \frac{1 + 2\lambda}{3}f'f' = 0 \tag{36a}$$

subject to

$$f(0) = 0, \quad f''(0) = -1, \quad \text{and} \quad f'(\bar{\eta}) \rightarrow 0 \quad \text{as} \quad \bar{\eta} \rightarrow \infty. \tag{36b}$$

Results and discussion

The primary result of this paper is that the effect of longitudinal surfaces of the free convective boundary layer flow from a heated vertical surface in a porous medium can be described by means of the similarity solutions which apply for the equivalent plane surface configurations. Such a result is found to be valid for a surprisingly wide class of vertical free convection flows, namely, those for which (i) a prescribed power-law temperature is set on the surface, and (ii) a prescribed power-law heat flux is set. We note that our analysis also applies for inclined heated surfaces where the similarity variable has to be redefined in a straightforward way to take account of the reduced buoyancy force operating along the surface.

It is necessary to determine the range of values of x (or ξ) for which the present analysis is valid. For the case of a prescribed temperature of the surface, the term, $(1 + a^2 \cos^2 \hat{z})Ra \xi^{\lambda-1} \frac{\partial^2}{\partial \eta^2}$, was assumed to be the largest term in the definition of the operator, \mathcal{L} , given in (18). Therefore our analysis is valid if it is larger than both $\partial^2/\partial \xi^2$ and $\partial^2/\partial \hat{z}^2$ as $Ra \rightarrow \infty$. Therefore, when the surface has a constant temperature (i.e. when $\lambda = 0$), this means that our analysis is valid as long as $O(Ra^{-1}) \ll \xi \ll O(Ra)$. More generally, a detailed study of the orders of magnitude of the various terms shows that our analysis is valid whenever

$$O(Ra^{-1/(1+\lambda)}) \ll \xi \ll O(Ra^{1/(1-\lambda)}) \quad \text{when } -1 < \lambda < 1$$

and

$$O(Ra^{-1/(1+\lambda)}) \ll \xi \quad \text{when } 1 < \lambda.$$

Whenever $\lambda < -1$ our analysis does not apply. When the surface has a prescribed rate of heat flux a similar study shows that ξ must satisfy the asymptotic relations,

$$O(Ra^{-1/(2+\lambda)}) \ll \xi \ll O(Ra^{1/(1-\lambda)}) \quad \text{when } -2 < \lambda < 1$$

and

$$O(Ra^{-1/(2+\lambda)}) \ll \xi \quad \text{when } 1 < \lambda.$$

Again, when $\lambda < -2$ our analysis is invalid. For all valid cases (i.e. when $\lambda > -1$ for a prescribed surface temperature and when $\lambda > -2$ for a prescribed surface heat flux) the $\xi = O(1)$ range of values admits similarity solutions.

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