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THE EFFECT OF INERTIA ON THE ONSET OF MIXED CONVECTION IN A POROUS LAYER HEATED FROM BELOW

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ABSTRACT

A linear stability analysis is presented which determines how the presence of fluid inertia and a mean applied horizontal pressure gradient modifies the familiar Darcy–flow criterion for the onset of convection in the porous–Bénard problem. It is found that, when both effects are present, the critical Rayleigh number is raised above $4\pi^2$, and increasingly so as either the strength of inertia effects or the magnitude of the pressure gradient or both increase. In general, spanwise periodic vortices are favoured with no change in the critical wavenumber. However, when the flow is restricted to being two–dimensional, the critical wavenumber is also modified. The result of this analysis provides a straightforward qualitative and quantitative means whereby experimental tests can be used to validate or otherwise the Forchheimer model of the effect of fluid inertia.

Introduction

In this note we consider the onset of mixed convection in a horizontal porous layer heated uniformly from below in the presence of inertial effects as modelled by Forchheimer's extension to Darcy's law [1]. By mixed convection, it is meant that there is an overall horizontal fluid motion caused by an externally imposed horizontal pressure gradient, and therefore the basic state we analyse consists of a horizontal plug flow and a linear temperature drop across the layer. In the absence of inertia, Prats [2] showed that, for two–dimensional flows, the criterion governing the onset of convection is unchanged from the classical no–flow case considered by Lapwood [3] and Horton & Rogers [4], although the cellular pattern moves at the same speed as the horizontal flow. In the absence of the horizontal pressure gradient, the onset criterion is also unchanged when Forchheimer's quadratic extension to Darcy's law is assumed (see Nield & Joseph [5], He & Georgiadis [6] and Rees [7]), although the post critical behaviour is modifed.

When both effects are present the critical Rayleigh is changed, and, depending on the orientation of the rolls, so is the critical wavenumber — this is the subject of the present note. The value of the critical Rayleigh number depends on the orientation of the roll relative to the direction of flow. In an unbounded layer, roll axes are aligned with the flow with no change in the critical wavenumber, but these conclusions do not hold in suitably confined layers. As the critical Rayleigh number increases monotonically, and almost linearly, with increasing flow rate, this problem is clearly one which may be used to determine experimentally whether or not the Forchheimer quadratic drag term is appropriate for any chosen porous medium.

Governing equations and two–dimensional stability analysis

The nondimensional equations governing free convection flow in a porous medium are given by

$$u_x + v_y + w_z = 0, \tag{1a}$$

$$u(1 + Gq) = -p_x, \tag{1b}$$

$$v(1 + Gq) = -p_y + R\theta, \tag{1c}$$

$$w(1 + Gq) = -p_z, \tag{1d}$$

$$\theta_t + u\theta_x + v\theta_y + w\theta_z = \theta_{xx} + \theta_{yy} + \theta_{zz}, \tag{1e}$$

where $R = \rho g \hat{\beta} d K \Delta T / \mu / \kappa$ is the Darcy–Rayleigh number based on ρ, the reference density; g, gravity; $\hat{\beta}$, the coefficient of cubical expansion; d, the depth of the layer; K, the permeability of the medium; ΔT, the temperature drop across the layer; μ, the fluid viscosity; and κ, the thermal diffusivity. In equations (1b), (1c) and (1d) the fluid flux speed, q, is given by

$$q^2 = u^2 + v^2 + w^2, \tag{2}$$

where q is taken to be positive, and the inertia parameter, G, is given by

$$G = \frac{\tilde{K}\rho\kappa}{\mu d}. \tag{3}$$

Values of K and the material parameter, $\tilde{K}$, are given by Ergun's experimental relations [8],

$$K = \frac{L^2\Phi^3}{150(1-\Phi)^2}, \qquad \tilde{K} = \frac{1.75L}{150(1-\Phi)}, \tag{4}$$

where L is a characteristic particle or pore diameter and Φ denotes the porosity of the medium. In deriving equations (1) the Boussinesq approximation has been invoked, and it is assumed that the velocity field adjusts instantaneously to changes in the pressure and temperature.

When considering two–dimensional flows, we introduce the streamfunction, ψ, according to

$$u = \psi_y, \qquad v = -\psi_x, \qquad w = 0, \tag{5}$$

and hence equations (1) reduce to

$$(1 + Gq)(\psi_{xx} + \psi_{yy}) + G(\psi_x q_x + \psi_y q_y) = R\theta_x, \tag{6a}$$

$$\theta_t + \psi_x \theta_y - \psi_y \theta_x = \theta_{xx} + \theta_{yy}. \tag{6b}$$

where the fluid flux magnitude, q, a positive quantity, is given by

$$q^2 = u^2 + v^2 + w^2 = \psi_x^2 + \psi_y^2. \tag{6c}$$

The basic flow we shall consider is given by

$$\psi = Qy, \qquad \theta = 1 - y, \tag{7}$$

and therefore we solve equations (6a,b) subject to the boundary conditions,

$$y = 0: \qquad \theta = 0, \qquad \psi = 0 \tag{8a}$$

$$y = 1: \qquad \theta = 0, \qquad \psi = Q \tag{8b}$$

This basic profile which we shall analyse for stability corresponds to a linear temperature drop across the layer and a uniform horizontal flow induced by a constant applied horizontal pressure gradient. The flow is in the positive x–direction when Q is positive. It proves convenient to work in terms of the flow rate, Q, which corresponds to an imposed horizontal pressure gradient, $-Q(1 + G|Q|)$, than to specify the gradient and determine the flow rate.

If we now subtract out the basic solution, (7), using

$$\psi = \Psi + Qy, \qquad \theta = \Theta + 1 - y, \tag{9a, b}$$

and linearise the resulting equations, then the linear stability equations are

$$(1 + G|Q|)\nabla^2\Psi + G|Q|\Psi_{yy} = R\Theta_x, \tag{10a}$$

$$\nabla^2\Theta = \Theta_t - \Psi_x + Q\Theta_x \tag{10b}$$

and the boundary conditions are that $\Psi = \Theta = 0$ at both $y = 0$ and $y = 1$. The detailed derivation of (10) uses the fact that

$$q = |Q| - \text{sign}(Q)\Psi_y + \text{ higher order terms;} \tag{11}$$

this expression is precise except in the very near neighbourhood of those points where $\Psi_y = 0$. As such regions are asymptotically small — for this is a linear stability analysis — we may neglect their presence. However, should the present analysis be extended to a consideration of weakly nonlinear stability, then it may be necessary to account for these regions.

It is now a very straightforward matter to determine the critical Rayleigh number above which disturbances grow. On setting,

$$\Psi \quad \propto \quad e^{\lambda t} \sin k(x - Qt) \sin \pi y, \tag{12a}$$

$$\Theta \quad \propto \quad e^{\lambda t} \cos k(x - Qt) \sin \pi y, \tag{12b}$$

we find that neutrality corresponds to $\lambda = 0$ and that the critical Rayleigh number corresponding to a cell of wavenumber, k, is

$$R = \frac{(\pi^2 + k^2)^2 + G|Q|(\pi^2 + k^2)(2\pi^2 + k^2)}{k^2}. \tag{13}$$

Clearly the immediate consequence of this result is that the combined presence of inertia and a mean horizontal pressure gradient serves to raise the critical Rayleigh number for any given wavenumber. All neutral modes move with speed Q, a result which is identical to that of Prats [2] for Darcy flow. The critical wavenumber is found by minimising R over k, and it is given by

$$k_c = \pi\Big(\frac{1 + 2G|Q|}{1 + G|Q|}\Big)^{1/4}. \tag{14a}$$

The critical Rayleigh number may be written in the form,

$$R_c = \pi^2[(1 + G|Q|)^{1/2} + (1 + 2G|Q|)^{1/2}]^2/ \tag{14b}$$

vskip 10pt When GQ is small, we obtain

$$k_c \sim \pi(1 + \frac{1}{4}G|Q|) \qquad \text{and} R_c \sim \pi^2(4 + 6G|Q|), \tag{15a, b}$$

and therefore in this limit we recover the Darcy–flow results. But when GQ is large, then

$$k_c \sim 2^{1/4}\pi \qquad \text{and} \qquad R_c \sim \pi^2(3 + 2\sqrt{2})G|Q|. \tag{16a, b}$$

When GQ is nonzero convection cells at onset do not have a square aspect ratio, and it is clear that convection may be deferred to increasingly high Rayleigh numbers simply increasing the net flow along the layer. We also note that the slope of R_c against $G|Q|$ changes vary little as $G|Q|$ varies; it is close to $6\pi^2$ when $G|Q|$ is small, and decreases to $(3+2\sqrt{2})\pi^2 \simeq 5.828\pi^2$ when $G|Q|$ is large. The resulting graph of R_c against $G|Q|$ does not therefore deviate greatly from that of a straight line, and therefore we do not present it here.

Onset of rolls at other orientations

For rolls whose axes are at an arbitrary inclination to the direction of the mean flow the overall fluid motion and temperature field must necessarily be three–dimensional. Therefore we return to the primitive variables equations given in (1). The basic flow about which we perturb is given by

$$u = Q, \quad v = 0, \quad w = 0, \quad \theta = 1 - y, \quad \text{and} \quad p_x = -Q(1 + G|Q|). \tag{17a–e}$$

Denoting the perturbations to fluid velocities, the temperature and pressure by their respective upper–case letters, we obtain the following system of equations for the linearised perturbations,

$$U_x + V_y + W_z = 0 \tag{18a}$$

$$U(1 + 2G|Q|) = -P_x \tag{18b}$$

$$V(1 + G|Q|) = -P_y + R\Theta \tag{18c}$$

$$W(1 + G|Q|) = -P_z \tag{18d}$$

$$\Theta_t + Q\Theta_x - v = \Theta_{xx} + \Theta_{yy} + \Theta_{zz}. \tag{18e}$$

These equations admit solutions of the form,

$$\begin{pmatrix} U \\ V \\ W \\ \Theta \\ P \end{pmatrix} = \exp[\lambda t + ik((x - Qt)\cos\alpha + z\sin\alpha)] \begin{pmatrix} U^* \cos \pi y \\ V^* \sin \pi y \\ W^* \cos \pi y \\ \Theta^* \sin \pi y \\ P^* \cos \pi y \end{pmatrix} \tag{19}$$

where α is the inclination of the roll away from the direction of mean flow, and the starred quantities appearing on the right hand side of the equation are constants. Once more neutral stability corresponds to when $\lambda = 0$, and we find that the critical Rayleigh number for given values of α and k is

$$R = (\pi^2 + k^2)(1 + G|Q|) + \frac{\pi^2(\pi^2 + k^2)}{k^2\left[\frac{\cos^2\alpha}{1+2G|Q|} + \frac{\sin^2\alpha}{1+G|Q|}\right]}. \tag{20}$$

For general orientations of the roll axes the minimising value of the wavenumber is

$$k_c = \pi\left[\frac{1 + G|Q|}{1 + 2G|Q|}\cos^2\alpha + \sin^2\alpha\right]^{-1/4} \tag{21}$$

and the corresponding critical Rayleigh number is

$$R_c = \pi^2(1 + G|Q|)\Big[1 + \frac{[1 + 2G|Q|]^{1/2}}{[1 + GQ(1 + \sin^2\alpha)]^{1/2}}\Big]^2. \tag{22}$$

When $\alpha = \pm\pi/2$ we have the smallest possible value of the critical Rayleigh number,

$$R_c = 4\pi^2(1 + G|Q|) \tag{23}$$

and $k_c = \pi$. The two-dimensional results of the last section are recovered when $\alpha = 0$.

Conclusions

In this note we have sought to determine how the combined presence of fluid inertia, as modelled by the quadratic Forchheimer terms, and a horizontal mean flow serve to modify the criterion for the onset of cellular convection in a horizontal porous layer heated from below. Given that both these effects appear in the results quoted for the critical Rayleigh number and wavenumber only in terms of the product, GQ, we see that the presence of both is essential to modify the stability criterion.

We have seen that the preferred axial orientation is in the direction of the mean flow if the layer is unbounded in horizontal extent. In these circumstances the critical wavenumber is unmodified from the free convection Darcy-flow case, but the critical Rayleigh number rises linearly with $G|Q|$. When the layer is sufficiently restricted spatially to allow only two-dimensional motion, the critical wavenumber also increases with increasing values of $G|Q|$, but it is bounded above by the value $2^{1/4}\pi$.

From an experimental point of view, the results we present provide a relatively straightforward quantitative means whereby the validity of the Forchheimer model of inertia effects may be tested. This is especially so for media where inertia effects are relatively weak, for in these cases an increased flow rate through the medium will raise the critical Rayleigh number to levels which are easily distinguished from the Darcy-flow value.

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