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The stability of vertical thermal boundary-layer flow in a porous medium

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ABSTRACT. – We consider the linearised stability characteristics of the thermal boundary layer induced by the uniform heating of a semi-infinite vertical surface embedded in a fluid – saturated porous medium. In this paper attention is restricted to two-dimensional disturbances far from the leading edge. This analysis complements and extends the direct numerical simulation of Rees [1993] which shows that the flow is stable at locations sufficiently close to the leading edge. In this asymptotic regime we also find that wave disturbances decay. However, the rate of decay decreases as the distance downstream of the leading edge increases.

1. Introduction

The subject of thermal convection in porous media has attracted considerable attention in the last twenty years and is now considered to be an important field of study in the general areas of fluid dynamics and heat transfer. The growing volume of work devoted to this topic is amply documented by the most recent reviews: Nield [1984], Tien & Vafai [1989] and Nield & Bejan [1992]. There are many important reasons for this development. On the practical side, there is much interest in a new generation of engineering problems connected with the topical issues of thermal insulation engineering, packed-bed catalytic reactors, and heat storage beds, and, on the theoretical side, there remains a continuing need for a comprehensive theoretical framework which covers the field in much the same way as the solutions of the Navier-Stokes and energy conservation equations cover thermal convection in fluids.

Many papers dealing with convective flows in porous media have been motivated by geothermal applications. The formation of geothermal reservoirs is believed to be associated with the presence of recent volcanism or intense tectonic movements. Such activity can result in, for example, (i) the production of magmatic intrusions in subterranean aquifers, which cool to form vertical or near-vertical impermeable dikes, and (ii) the formation of a large region of heated bedrock [Cheng, 1978]. These systems are typically idealised in the first instance by considering the heated surface to be flat and semi-infinite – the former being either vertically aligned or inclined, the latter, horizontal. These idealised flows have been analysed using boundary layer theory and

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can be described in terms of similarity solutions ([Cheng & Minkowycz, 1977]; [Cheng & Chang, 1976]). This work has also been extended using higher-order boundary layer theory and the method of matched asymptotic expansions in order to obtain more accurate estimates of the heat transferred into the medium [Cheng & Hsu, 1984]; [Chang & Cheng, 1983].

The successful modelling of geothermal reservoirs relies on accurate accounts of the flow and heat transfer within them. It is therefore of considerable importance to investigate the stability of the thermally induced boundary layer flows since the change in the flow-field caused by an instability will alter substantially the rate of heat transfer. The aim of this paper is to investigate the stability of the thermal boundary layer flow induced by heating uniformly a semi-infinite vertical surface.

Few papers have been published which deal with thermal boundary layer instabilities in porous media, and we shall review them very briefly here. Most of these papers concentrate on vortex disturbances within either an inclined or a horizontal boundary layer ([Hsu, Cheng & Homsy, 1978]; [Hsu & Cheng, 1979] and [Jang & Chang, 1989]). It is found that vortex disturbances appear at increasing distances from the leading edge as the inclination from the horizontal increases. A recent paper by Rees & Bassom [1994] on the horizontal thermal boundary layer has shown that two-dimensional waves are destabilised closer to the leading edge than are vortices: such a result is contrary to what is generally assumed to be the case for this boundary layer. A more recent paper by Storesletten and Rees [1994] has attempted to investigate more accurately the criterion for instability by including the effects of higher order terms in the asymptotic expansion for the basic boundary layer. The conclusions from their studies are that (i) for a horizontal thermal boundary layer the inclusion of higher-order effects are insufficient to obtain accurate estimations of the point of neutral stability; in some cases the wavespeed of two-dimensional wave disturbances are negative, and (ii) for an inclined surface, accurate results are obtained only when the heated surface is within 5° from the vertical.

However, it must be borne in mind that all the analyses quoted in the above paragraph are linearised analyses based on the parallel-flow approximation; nonlinear and nonparallel effects may render these conclusions unreliable. The study by Rees [1993] attempted to circumvent the possible problems with nonparallelism by performing a direct numerical simulation of the evolution of two-dimensional wave disturbances in the vertical thermal boundary layer. Within the restriction of a necessarily finite computational domain he found that wave disturbances always appear to decay. In this paper we extend Rees's computational work by considering the wave stability characteristics at asymptotically large distances from the leading edge. It is found that the disturbance is concentrated in a narrow region adjacent to the heated surface and well within the basic boundary layer, thus giving a two-layer structure. The disturbances are found to decay but decreasingly so as the distance downstream increases. This analysis, which is presented in §3, is preceded by a study of the analogous vertical channel problem originally considered by Gill [1969]. Gill showed that small disturbances decay for all values of the Darcy-Rayleigh number; §2 quantifies Gill's result and provides substantial information for the detailed disturbance structure for the following boundary layer problem. We discuss the results briefly in §4.

2. The channel problem

In this section we examine the stability of wave-like modes in a flow field confined between two isothermal impermeable vertical walls. We choose a rectangular Cartesian co-ordinate system $(x,\ y,\ z)$ aligned so that the y-axis is perpendicular to the walls and Ox points vertically upwards. If all the distances are non-dimensionalised on the half-distance between the isothermal surfaces then the governing Darcy-Boussinesq equations are given by

(1) $$\operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u} = -\nabla p + (R\theta,\ 0,\ 0), \qquad \nabla^2\theta = \theta_t + (\mathbf{u}\cdot\nabla)\,\theta,$$

where u, v, w are the velocity components in the three co-ordinate directions, $\mathbf{u} = (u,\ v,\ w)$ and p and θ are the dimensionless fluid pressure and temperature respectively. Furthermore, ∇^2 denotes the usual three-dimensional Laplacian operator and R is the Darcy-Rayleigh number of the flow. Under this nondimensionalisation the isothermal boundaries are given by $y = \pm 1$ and Eqs. (1) need to be solved subject to the boundary conditions

(2) $$v = 0, \qquad \theta = \pm 1, \quad \text{on } y = \pm 1.$$

It is clear that this system admits the simple solution

(3) $$v = w = 0, \qquad u = Ry, \qquad p = 0, \qquad \theta = y,$$

and in order to analyse the linear stability characteristics of this flow we perturb it by writing

(4) $$(u,\ v,\ w,\ p,\ \theta) = (Ry,\ 0,\ 0,\ 0,\ y) + \delta\,(\tilde{u},\ \tilde{v},\ \tilde{w},\ \tilde{p},\ \tilde{\theta}),$$

where δ is taken to be an infinitesimal quantity. The substitution of (4) into (1), followed by linearisation, yields

(5) $$\operatorname{div}\tilde{\mathbf{u}} = 0, \qquad \tilde{\mathbf{u}} = -\nabla\tilde{p} + (R\tilde{\theta},\ 0,\ 0), \qquad \nabla^2\tilde{\theta} = \tilde{\theta}_t + \tilde{v} + Ry\tilde{\theta}_x,$$

which needs to be solved subject to the boundary conditions

(6) $$\tilde{v} = \tilde{p}_y = 0 \quad \text{on } y = \pm 1.$$

It is easiest to proceed by eliminating $\tilde{u}$, $\tilde{w}$ and $\tilde{p}$ from (5) and, in order to examine the stability of wave-like modes which are periodic in both the x and z directions, we write

$(7a, b)$ $$\tilde{v} = i\alpha F(y)\, e^{i\,(lz+kx)+\lambda t}, \qquad \tilde{\theta} = G\,(y)\, e^{i\,(lz+kx)+\lambda t},$$

where both l and k are real valued and λ is, in general, complex. Eqs. (5) then reduce to the pair of ordinary differential equations,

$$F'' - \alpha^2 F + SG' = 0, \qquad G'' - (\alpha^2 + \lambda)\, G - i\alpha\,(F + SyG) = 0, \tag{8a, b}$$

where $\alpha = (l^2 + k^2)^{1/2}$ is the roll-wavenumber and $S = Rk/\alpha$. Notice that the number of parameters has been reduced by this transformation, which implies that Squire's theorem holds, and that the most unstable disturbances are therefore two-dimensional. Eqs. (8) were solved numerically subject to the boundary conditions

$$F = G = 0 \quad \text{on } y = \pm 1, \tag{9}$$

and this was accomplished using a multiple-shooting code coupled to a fourth-order Runge-Kutta scheme.

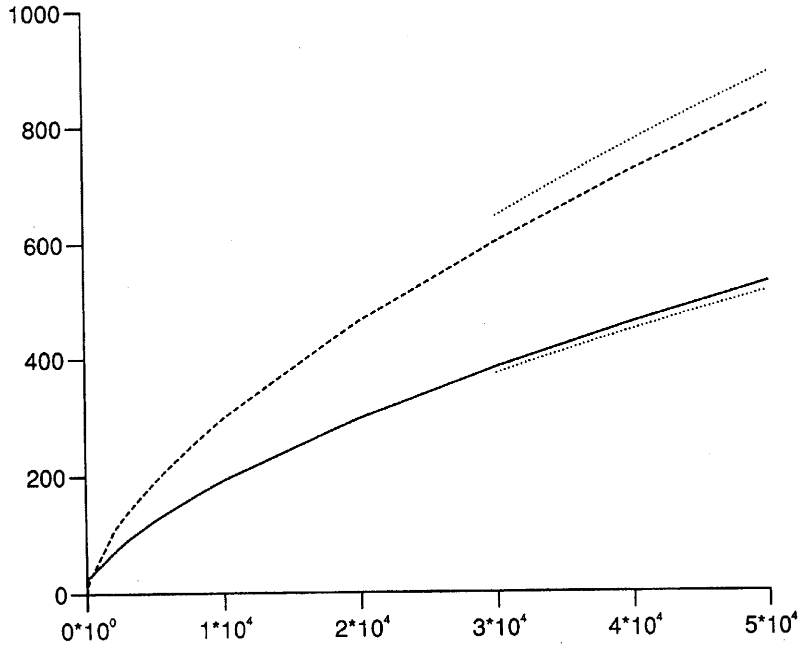


Fig. 1. – The variation of $(i\alpha S - \lambda)$ with the Darcy-Rayleigh number, S, for wave number $\alpha = \pi/10$; the solid and broken lines denote the real and the imaginary parts respectively. The dotted lines correspond to the respective asymptotic predictions.

In Figures 1 and 2 we present results corresponding to the wavenumber $\alpha = \pi/10$ and values of the modified Rayleigh number S up to 5×10^4, detailed results for low values of S (up to 200) are given in Rees [1988]. Figure 1 illustrates the dependence on S of the real and imaginary parts of the growth rate λ for the least stable mode. Figure 2 depicts the corresponding eigenfunctions F and G when $S = 5 \times 10^4$. Importantly, it is clear that, for this range of values of S, we have $Re\,(\lambda) < 0$ which confirms that the base flow (3) is stable to waves and is consistent with Gill's result. Gill's analysis, however, merely indicated the qualitative result that wave disturbances decay; he did not

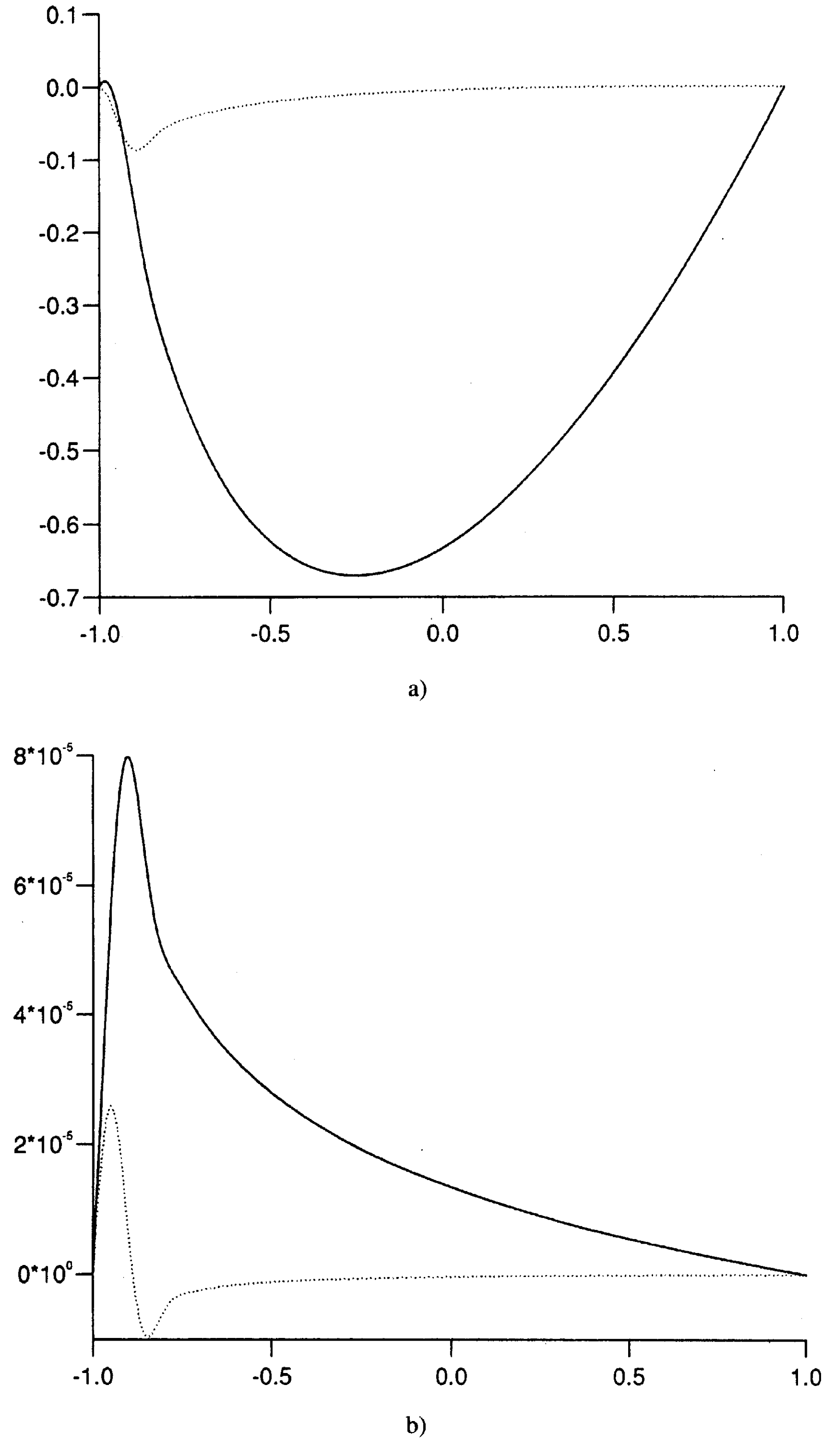


Fig. 2. – Forms of the eigenfunctions a) F and b) G when $S = 50\,000$. The real parts of the functions are denoted by solid lines and the imaginary parts by broken lines.

attempt to compute rates of decay as a function of the Rayleigh number. This we shall undertake by obtaining solutions of (8) and (9) in the asymptotic limit $S \to \infty$.

To begin this analysis it is worthwhile to recall the forms of the eigenfunctions for large values of S; see *Fig.* 2. These indicate that, although F and G vary smoothly and slowly across the majority of the of the region $-1 < y \leq 1$, a well-defined boundary-layer structure is apparent next to $y = -1$. Across $-1 < y \leq 1$ we take $G = O(1)$ and balance terms in (8a) and (8b), respectively, to give $F = O(S)$ and $\lambda = O(S)$. If we write

$$F = S\hat{f}_0(y) + \ldots, \qquad G = \hat{g}_0(y) + \ldots, \qquad \lambda = S\lambda_0 + \ldots, \tag{10}$$

Eqs. (8) yield

$$\hat{f}_0'' - \alpha^2 \hat{f}_0 + \hat{g}_0' = 0, \qquad \hat{f}_0 = -(y + \lambda_0/i\alpha)\,\hat{g}_0. \tag{11a, b}$$

Eliminating $\hat{f}_0$ between this pair gives

$$(y - i\lambda_0/\alpha)\,\hat{g}_0'' + \hat{g}_0' - \alpha^2 (y - i\lambda_0/\alpha)\,\hat{g}_0 = 0 \tag{12}$$

and the solution of this equation which satisfies the requirement $\hat{g}_0(1) = 0$ is

$$\hat{g}_0 = B\left[K_0(\zeta) - \frac{K_0(\alpha - i\lambda_0)}{I_0(\alpha - i\lambda_0)}\,I_0(\zeta)\right], \tag{13}$$

where $\zeta \equiv \alpha y - i\lambda_0$, K_0 and I_0 are modified Bessel functions of order zero (*see* Abramowitz & Stegun [1965]), and B is a normalising constant. It is easily shown that $\hat{g}_0$ does not tend to zero as $y \to -1$ and it follows, from (11b), that $\hat{f}_0 \to 0$ in this limit only if $\lambda_0 = i\alpha$. Under this assumption it is a straightforward exercise to prove that the boundary layer next to $y = -1$ is of depth $O(S^{-\frac{1}{3}})$ and, furthermore, if within this layer $y = -1 + S^{-\frac{1}{3}}\,Y$, where $Y = O(1)$, then

$$G \to -B \ln S^{-\frac{1}{3}} - B\left[\ln\left(\frac{\alpha Y}{2}\right) + \gamma + \frac{K_0(2\alpha)}{I_0(2\alpha)}\right] + \cdots \quad \text{as } Y \to \infty, \tag{14}$$

in order to match with (13). In this form, γ denotes Euler's constant [A & S, 1965] and the eigenfunctions expand as

$$F = S^{\frac{2}{3}}(\ln S^{-\frac{1}{3}})\,f_0(Y) + S^{\frac{2}{3}}\,f_1(Y) + \cdots, \tag{15a}$$

$$G = (\ln S^{-\frac{1}{3}})\,g_0(Y) + g_1(Y) + \cdots; \tag{15b}$$

this suggests that the growth rate of the wave takes the form

$$\lambda = i\alpha S + S^{\frac{2}{3}}\,\lambda_1 + S^{\frac{2}{3}}(\ln S^{-\frac{1}{3}})^{-1}\,\lambda_2 + \cdots \tag{15c}$$

Expansions (15) and Eqs. (8) yield

$$f_0'' + g_0' = 0, \qquad g_0'' - (\lambda_1 + i\alpha Y)\,g_0 - i\alpha\,f_0 = 0, \tag{16a, b}$$

whence

$$(16c) \qquad g_0 = C \int_{\tau_0}^{\tau} Ai\,(s)\, ds, \qquad \tau = -\tau_0 + (i\alpha)^{\frac{1}{3}}\, Y, \qquad \tau_0 \equiv \lambda_1\,(i\alpha)^{-\frac{2}{3}},$$

with Ai denoting the usual Airy function. (We remark that the constant C may be found in terms of B by matching but details of this are not needed here.) Solution (16c) satisfies $g_0\,(0) = 0$; f_0 now follows from (16b) and the boundary condition $f_0\,(0) = 0$ forces

$$(17) \qquad Ai'\,(\tau_0) = 0.$$

This equation admits an infinity of roots and these all lie on the negative real axis. The first solution, $\tau_0 \sim -1.019$, corresponds to the least stable wave-like mode and then (16c) gives

$$(18) \qquad \lambda_1 = -1.019\alpha^{\frac{2}{3}}\,(1 + i\sqrt{3})/2.$$

Since $\lambda_0 = i\alpha$ is purely imaginary we conclude from (18) that it is the $O\,(S^{\frac{2}{3}})$ term in expansion (15c) for λ which has the largest non-zero real part. Further, as this real component is negative, the disturbances decay with time and stability is implied in the limit $S \to \infty$.

The two-term asymptotic expansion for λ is found to be not particularly accurate when compared with the numerical results for S up to 5×10^4. The principal reason for this follows from the form of (15c) which demonstrates that, unless S is extremely large indeed, then the second and third terms in this expansion are of roughly comparable magnitudes. In order to improve the agreement between the asymptotic and numerical results it is therefore worthwile seeking the value of λ_2 in (15c). It is found that

$$(19a, b) \qquad g_1'' - (\lambda_1 + i\alpha Y)\, g_1 - i\alpha f_1 = \lambda_2 g_0, \qquad f_1'' + g_1' = 0,$$

and the second of these is readily integrated once with the constant of integration determined by matching with the core region solutions, $\hat{f}_0$ and $\hat{g}_0$. Elimination of f_1 from (19a) then gives a inhomogeneous third-order equation for g_1 which, when solved subject to the boundary conditions $f_1 = g_1 = 0$ on $Y = 0$, gives that

$$(20) \qquad \lambda_2 = -\frac{(i\alpha)^{\frac{2}{3}}}{\tau_0\,(Ai\,(\tau_0))^2} \left(\int_{\tau_0}^{\tau} Ai\,(s)\, ds \right)^2 \simeq 2.239\alpha^{\frac{2}{3}}\,(1 + i\sqrt{3})/2.$$

The difference between the numerical results and the asymptotic predictions for λ are much reduced by the inclusion of λ_2. For a given large S, it is the least stable mode for which the agreement between asymptotic and numerical values is best and this agreement improves with S; *see* Table I, in which the subscripts N and A on values of

TABLE I. – First mode comparisons of asymptotic and numerical results for $\alpha = \pi/10$

S	Re (λ_N)	Re (λ_A)	Rel. error	Im (λ_N)	Im (λ_A)	Rel. error
200	−17.534	−18.071	0.031	37.778	31.532	0.165
2×10^4	−296.99	−288.94	0.027	5 819.7	5 782.7	0.006
5×10^4	−529.63	−514.21	0.029	14 873	14 817	0.004

the eigenvalue λ denote values which have been obtained numerically, or by using the asymptotic form (15c), respectively.

Results (18) and (20) prove that, as $S \to \infty$, then

$$\mathrm{Re}\,(\lambda) = (\alpha S)^{\frac{2}{3}} \left[-0.56 + \frac{1.12}{\ln\, S^{-\frac{1}{3}}} + \cdots\right],$$

so that the decay rate of the modes is seen to increase both with S and with the wavenumber α.

3. The boundary layer problem

In this section we consider the stability of wave-like modes in the thermal boundary layer flow induced by a single heated vertical surface. We note that, although Squire's thereom does not apply to this problem, the resulting three-dimensional disturbance equations have the same structure as the two-dimensional equations presented below, and the qualitative nature of the solutions are unchanged. Thus we concentrate on two-dimensional modes for the sake of brevity.

The nondimensional equations governing unsteady two-dimensional Darcy-Boussinesq convection in a porous medium are

$$(21a, b) \qquad \Psi_{xx} + \Psi_{yy} = \Theta_y, \qquad \Theta_t = \nabla^2\Theta + \Psi_x\Theta_y - \Psi_y\Theta_x,$$

where the x-axis is again orientated vertically and Ψ and Θ are the dimensionless streamfunction and temperature respectively. Here ∇^2 denotes the two-dimensional Laplacian operator and the Darcy-Rayleigh number has been scaled out of the equations as there is no natural length scale in this problem (*see* Riley and Rees [1985] for details of the nondimensionalisation). The boundary conditions required to complete the specification of the problem are the $\Psi = 0$ and $\Theta = 1$ on the positive x-axis, $\Psi = 0$ and $\Theta_y = 0$ on the negative x-axis and that the ambient temperature tends to zero as $x \to \infty$. In this Cartesian co-ordinate system it is well-known that the vertical boundary layer thickness increases with x, and is asymptotically proportional to $x^{\frac{1}{2}}$. The introduction of parabolic co-ordinates defined by

$$x = \frac{1}{4}\,(\sigma^2 - \eta^2), \qquad y = \frac{1}{2}\,\sigma\eta,$$

yields the equations

$$\text{(22}a,b\text{)}\qquad \begin{cases} \Psi_{\sigma\sigma} + \Psi_{\eta\eta} = \frac{1}{2}\,[\sigma\Theta_\eta + \eta\Theta_\sigma], \\ \Theta_t = \frac{4}{\sigma^2+\eta^2}\,[\Theta_{\sigma\sigma} + \Theta_{\eta\eta} + \Psi_\sigma\Theta_\eta - \Psi_\eta\Theta_\sigma]. \end{cases}$$

Rees and Bassom [1991] demonstrated that this pair are satisfied exactly by the forms

$$\text{(23)}\qquad \Psi = \Psi_B = \frac{1}{2}\,\sigma\, f\,(\eta), \qquad \Theta = \Theta_B = g\,(\eta),$$

where $f\,(\eta)$ and $g\,(\eta)$ are defined by

$$\text{(24}a,b\text{)}\qquad f'' - g' = 0, \qquad g'' + \frac{1}{2}\, f g' = 0,$$

subject to

$$\text{(25)}\qquad f\,(0) = 0, \qquad g\,(0) = 1 \qquad \text{and} \qquad f',\, g \to 0 \quad \text{as } \eta \to \infty.$$

These ordinary differential equations were first solved in the context of convection in porous media by C & M [1977]. In order to analyse the linear stability characteristics of this flow we invoke the parallel-flow approximation (but see the comments in §4) and perturb (23) by writing

$$\text{(26)}\qquad (\Psi,\, \Theta) = (\Psi_B,\, \Theta_B) + \Delta\,\{\psi\,(\eta),\, \theta\,(\eta)\}\ \exp\,[i\alpha\sigma + \lambda t],$$

where Δ is taken to be infinitesimally small. Substitution of (26) into (22) followed by linearisation yields

$$\text{(27}a,b\text{)}\qquad \begin{cases} \psi'' - \alpha^2\psi = \frac{1}{2}\,[\sigma\theta' + i\alpha\eta\theta], \\ \frac{1}{4}\,(\sigma^2+\eta^2)\,\lambda\theta = \left(\theta' - \alpha^2\theta + i\alpha g'\psi + \frac{1}{2}\, f\theta' - \frac{1}{2}\, i\alpha f'\theta\right), \end{cases}$$

where σ is now a parameter which plays a role similar to that of the Darcy-Rayleigh number in §2. Eqs. (27) are to be solved in the asymptotic limit $\sigma \to \infty$ subject to the boundary conditions

$$\text{(28)}\qquad \psi\,(0) = \theta\,(0) = 0 \qquad \text{and} \qquad \psi',\, \theta \to \theta \quad \text{as } \eta \to \infty.$$

We begin the analysis by taking $\theta = O(1)$ in (27). When σ is large an order-of-magnitude analysis of (27) suggests that $\psi = O\,(\sigma)$ and $\lambda = O\,(\sigma^{-1})$. If we write

$$\text{(29)}\qquad \theta = \theta_0\,(\eta) + \cdots, \qquad \psi = \sigma\psi_0\,(\eta) + \cdots, \qquad \lambda = \lambda_0/\sigma + \cdots,$$

then

$$(30a, b) \qquad \psi_0'' - \alpha^2 \psi_0 = \frac{1}{2}\,\theta_0', \qquad \theta_0 = \frac{2g'\psi_0}{(g + \lambda_0/2i\alpha)}$$

and the elimination of θ_0 gives

$$(31) \quad (g + \lambda_0/2i\alpha)\,\psi_0'' - g'\psi_0' - \left\{g'' - \frac{(g')^2}{(g + \lambda_0/2i\alpha)} + \alpha^2\,(g + \lambda_0/2i\alpha)\right\}\psi_0 = 0.$$

Although Eq. (31) has some similarities with (12), which relates to the channel problem, it appears that is does not admit an obvious closed-form solution. In order that $\psi_0 \to 0$ as $\eta \to 0$, the form of Eq. (30*b*) suggests that we need $\lambda_0 = -2i\alpha$ which shows that ψ_0 can be taken to be real-valued throughout the zone $0 \leq \eta \leq \infty$. The numerical solution of (31) subject to the appropriate boundary conditions yields the eigenfunction which is shown in Figure 3*a* and Figure 3*b* shows results for the θ_0 given by (30*b*). It is easy to prove that, as $\eta \to 0$, we have

$$(32a, b) \qquad \psi \to (\sigma\omega \ln\,\omega)\,\hat{\eta} + \sigma\omega\hat{\eta}\ln\,\hat{\eta}, \qquad \theta \to 2\ln\,\omega + 2\ln\,\hat{\eta},$$

where we have taken $\eta = \omega\hat{\eta}$, $\hat{\eta} = O\,(1)$ and $\omega\,(\sigma) \ll 1$. In order to satisfy both the boundary conditions (28) at $\eta = 0$ it is necessary to introduce a boundary-layer type structure and it is a straightforward exercise to prove that this layer has depth $O(\sigma^{-\frac{1}{3}})$ and is located immediately adjacent to the heated wall. Therefore we set $\omega = \sigma^{-\frac{1}{3}}$ and take

$$(33a) \qquad \eta = \sigma^{-\frac{1}{3}}\,\hat{\eta}.$$

The eigenfunctions now expand as

$$(33b, c) \quad \psi = \sigma^{\frac{2}{3}}\,(\ln\sigma^{-\frac{1}{3}})\,\hat{\psi}_0\,(\hat{\eta}) + \sigma^{\frac{2}{3}}\,\hat{\psi}_1\,(\hat{\eta}) + \cdots, \qquad \theta = (\ln\sigma^{-\frac{1}{3}})\,\hat{\theta}_0\,(\hat{\eta}) + \hat{\theta}_1 + \cdots,$$

whilst the growth rate of the wave takes the form

$$(33d) \qquad \lambda = -\frac{2i\alpha}{\sigma} + \frac{\lambda_1}{\sigma^{\frac{4}{3}}} + \frac{\lambda_2}{\sigma^{\frac{4}{3}}\,(\ln\,\sigma^{-\frac{1}{3}})} + \cdots$$

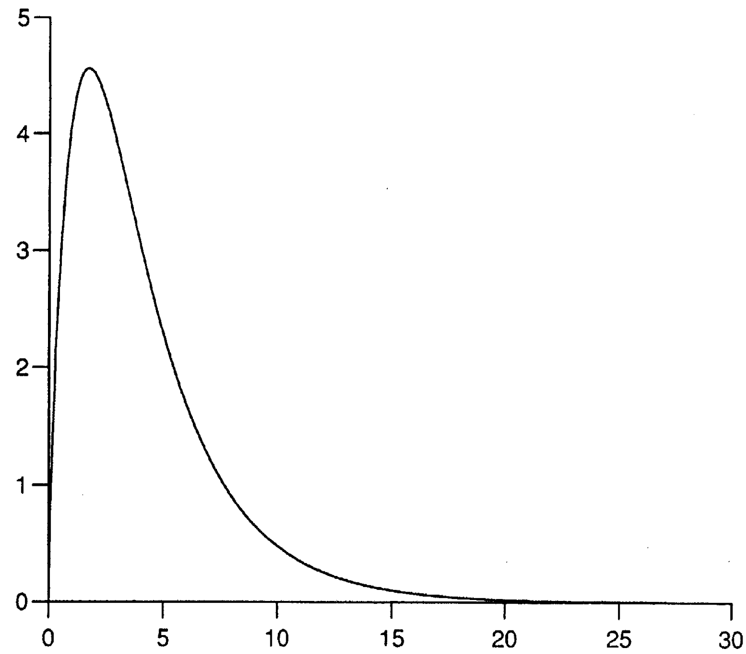
Substitution of (33) into (27) gives

$$(34a, b) \qquad \hat{\psi}_0'' - \frac{1}{2}\,\hat{\theta}_0' = 0, \qquad \hat{\theta}_0'' - \frac{1}{4}\,(\lambda_1 + 2i\alpha\hat{\eta}g_0')\,\hat{\theta}_0 + i\alpha g_0'\hat{\psi}_0 = 0,$$

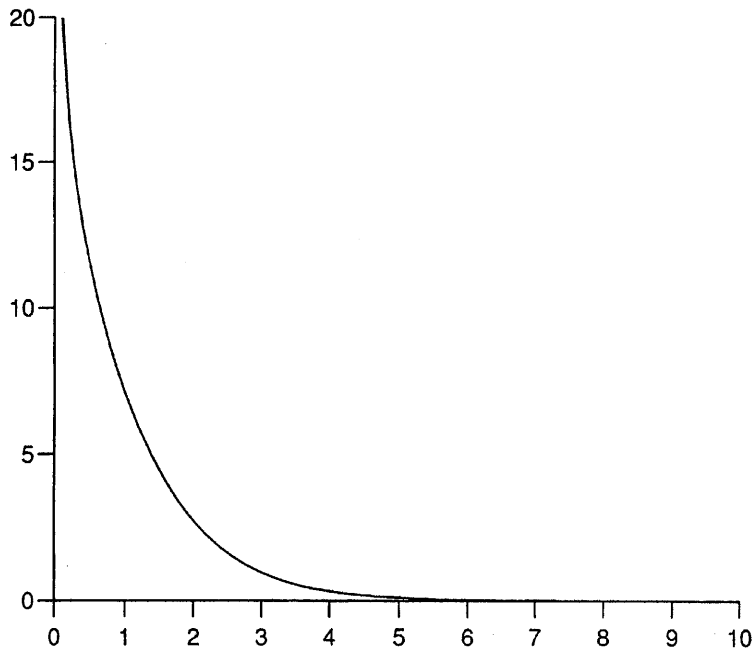
whence

$$(34c) \qquad \theta_0 = D\int_{\mu_0}^{\mu} Ai\,(s)\,ds, \qquad \mu = \left(\frac{i\alpha g_0'}{2}\right)^{\frac{1}{3}}\left(\hat{\eta} + \frac{\lambda_1}{2i\alpha g_0'}\right),$$

with $\mu_0 = \mu|_{\hat{\eta}=0}$ and D is a constant which can be determined by matching with (32). Solution (34*c*) automatically satisfies $\theta_0\,(0) = 0$ and substitution of this into (34*b*)



a)



b)

Fig. 3. – Leading order eigenfunctions a) ψ_0 and b) θ_0 as given by the numerical solution of Eqs. (31) and (30*b*).

yields $\hat{\psi}_0$. The boundary condition $\hat{\psi}_0\,(0) = 0$ means that $Ai'\,(\mu_0) = 0$ and, as in our discussion of (17), the first solution $\mu_0 \sim -1.019$ corresponds to the least stable wave-like mode. Then

$$\lambda_1 = -1.46\alpha^{\frac{2}{3}}\,(1 - i\sqrt{3})/2, \tag{35}$$

so that, just as for the channel problem of §2, λ_0 is purely imaginary and it is the λ_1 term in expansion (33d) which has the largest non-zero real part. This real component is negative and therefore linear stability is suggested in the limit $\sigma \to \infty$.

For completeness we quickly comment on the value of λ_2 in the growth rate expression (33d). It is routine to show that

$$\hat{\psi}_1'' - \frac{1}{2}\hat{\theta}_1' = 0, \qquad \hat{\theta}_1'' - \frac{1}{4}(\lambda_1 + 2i\alpha\hat{\eta})\hat{\theta}_1 - i\alpha g_0'\hat{\psi}_1 = \frac{1}{4}\lambda_2\hat{\theta}_0 \tag{36a, b}$$

and Eq. (36a) is readily integrated once and matched with the outer layer solutions. Elimination of $\hat{\psi}_1$ then gives a forced third-order equation for $\hat{\theta}_1$ and when the relevant boundary conditions are imposed we get

$$\lambda_2 = \frac{2^{4/3}(-\alpha g_0')^{2/3} e^{2\pi i/3}}{\mu_0}\left(\frac{\int_\infty^{\mu_0} Ai\,(s)\,ds}{Ai\,(\mu_0)}\right) \approx 3.22\alpha^{2/3}\,(1 - i\sqrt{3})/2. \tag{37}$$

Numerical solution of the fundamental Eqs. (27) as $\sigma \to \infty$ confirm this finding that stability is implied in this limit. Furthermore, reasonable agreement is found with the three-term asymptotic prediction which is constructed from (33d), (35) and (37). In the interest of brevity we do not detail the numerical and asymptotic comparisons as they are similar to those described for the channel problem of §2, but further details will eventually be found in Lewis's dissertation.

4. Discussion

In this paper we set out to examine the stability characteristics of the thermal boundary layer induced by uniform heating of a semi-infinite vertical surface embedded in a flluid-saturated porous medium. In §2 we re-examined the vertical channel problem of [G, 1969] which was shown to be stabe to all infinitesimal disturbances. The present analysis confirms this conclusion and quantifies it. We have shown that the basic flow is increasingly stable as the Darcy-Rayleigh number is increased for a fixed disturbance wavelength. In §3 we have extended the work of [R, 1993] who used a direct numerical simulation of evolving wave disturbances to show that the boundary layer is stable for disturbances sufficiently close to the leading edge of the boundary layer. This has been achieved by considering the wave stability characteristics asymptotically far from the leading edge. It is concluded that this flow is also stable, but is decreasingly so as the distance from the leading edge (which is analogous to the Darcy-Rayleigh number of the channel problem) increases. It is important to note that, although the parallel-flow approximation has been assumed for the analysis of §3, a formal nonparallel analysis similar to that of [Smith, 1979] for the Blasius boundary layer would be of purely theoretical rather of practical interest. The analysis of [S, 1979] is used to provide important information relating to precisely where a boundary layer flow is likely to become unstable, but further work on the present boundary layer problem would serve only to give a small correction to the already negative growth rate. The analysis we have described here has only been carried out to orders of magnitude that would be unaltered by inclusion of non-parallelism; consideration of this effect would only be worthwhile if extremely accurate decay rates were desired.

In conclusion, we remark that the fact that this particular boundary layer is stable makes it a somewhat novel flow. However, we also anticipate that thermal boundary layers on downward-facing inclined surfaces should also be stable. In view of the fact that the leading order term in (33*d*) is asymptotically small, we also suspect that even an asymptotically small tilt away from the vertical (resulting in an upward facing surface) might induce instability. A theoretical examination of this case is intended.

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