THE LINEAR VORTEX INSTABILITY OF FLOW INDUCED BY A HORIZONTAL HEATED SURFACE IN A POROUS MEDIUM

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SUMMARY

A linearized vortex instability theory for convection induced by a semi-infinite horizontal heated surface embedded in a fluid-saturated porous medium is developed. Due to the inadequacies of existing parallel-flow theories the problem has been re-examined using asymptotic techniques that use the distance downstream of the leading edge of the surface as the large parameter. The parallel-flow theories predict that at each downstream location there are two possible vortex wavenumbers which lead to neutrally stable modes. It is demonstrated how one of these disturbances is only weakly dependent on non-parallel terms, whereas the second mode is crucially dependent upon the non-parallelism within the flow. Consequently, this second mode cannot be described by any quasi-parallel approach and its properties may only be deduced by numerical computations of the full governing equations. We illustrate how our theory, which has similarities with that employed in the analysis of high wavenumber Görtler vortices in boundary layers above concave walls, may be used to isolate the most unstable vortex mode.

1. Introduction

THE topic of thermal convection in porous media has been one of considerable interest in the general areas of fluid dynamics and heat transfer. There are many reasons for this--on the practical side there is concern with a new generation of engineering projects dealing with topical issues like energy conservation and the optimization of heat transfer. Meanwhile on the theoretical front there is the need for a comprehensive framework which covers the field in much the same way as the solutions of the Navier–Stokes and energy-conservation equations cover thermal convection in fluids.

The study of convective instability in porous media has, with few exceptions, concentrated on basic flows that are confined in one or more coordinate directions and which are typically spatially uniform. For such problems a linear stability analysis is performed by appealing to the spatial uniformity in order to Fourier-decompose the disturbance into a sum of independent components

(1). The ordinary differential equations governing each of the disturbance components are solved to obtain the Darcy–Rayleigh number, say, as a function of the wavenumber of the disturbance. The spatial periodicity arising in the linear stability analysis is also used to extend the analysis into the nonlinear regime, although it is usual to have to employ numerical methods to carry this out (**2** to **5**). All the above-mentioned papers deal with the porous-medium analogue of the Bénard problem, the study of which has now reached well into the nonlinear regime with a detailed analysis of the transition to unsteady flow appearing only relatively recently (**6**).

A corresponding study of thermal boundary layers induced by semi-infinite heated surfaces embedded in a porous medium remains in its infancy, however. The principal reason for this is that the growing boundary layer is spatially non-uniform. As far as wave-like disturbances (or Tollmien–Schlichting waves) are concerned (where the resulting flow remains two-dimensional) a naive Fourier decomposition cannot be expected to give an accurate prediction of where the boundary layer becomes unstable because the basic flow is growing spatially and hence the disturbances will not be periodic. This type of porous-medium problem is of interest and has been motivated mainly by geothermal applications. The formation of geothermal reservoirs is thought to be associated with the presence of recent volcanism or intense tectonic movements. Such activity can result in the production of magmatic intrusions in subterranean aquifers which cool to form impermeable dikes or can lead to a large region of heated bedrock (**7**). These systems are often idealized in the first instance to be flat and semi-infinite surfaces embedded in a porous medium.

We have been unable to find a paper which deals with wave disturbances in porous-medium thermal boundary layers although there are a few analysing isothermal and non-isothermal boundary layers in fluids, see (**8** to **10**). A recent investigation by the present authors (**11**) has addressed the nonlinear evolution of wave disturbances in horizontal boundary-layer flow using numerical methods to solve the full governing equations. We found that the flow admitted several different nonlinear phenomena such as cell-merging, the eruption of plumes from the boundary layer, intermittent boundary-layer thinning and chaotic motion. The formation of plumes rising out of the boundary layer was seen to cause the temporary thinning of the boundary layer and the subsequent inhibition of instability near the leading edge. Thus there is ample evidence that an evolving instability influences conditions well upstream of itself, and we conclude that should a linear instability analysis be performed then its validity could, quite legitimately in view of the results of (**11**), be called into question.

In the present paper we analyse disturbances to the horizontal boundary layer which take the form of longitudinal vortices. Unlike the case for wave-like disturbances it is quite legitimate to Fourier-decompose the disturbances, but only in the spanwise direction. Hsu and Cheng (**12**) and Hsu, Cheng and Homsy (**13**) have performed linear stability analyses of convection from inclined and horizontal isothermal surfaces, respectively, using the parallel-flow approximation. This means that the vortex disturbances have a *prescribed* streamwise variation, thereby yielding a set of ordinary differential equations for the disturbances. The sole advantage of the approximation is that neutral curves may be calculated relatively easily, but it also has a major drawback. It has been known for some time that in non-parallel flows the linear stability properties of imposed disturbances are non-unique and depend to some extent on how and where the disturbance starts. This was probably first commented upon by Bouthier (**14**, **15**) for wave-like perturbations in non-parallel flow and, more recently, has been discussed extensively by Hall (**16**) in the context of Görtler vortex disturbances in boundary layers. The studies of Hall have shown that, except in the case of small spanwise vortex wavelength, the results of a parallel-flow analysis are completely unreliable for Görtler modes and this raises the possibility that a similar situation may occur for the vortex modes considered here. As further evidence of this it is worthwhile to examine the results of the parallel analysis for the thermal boundary-layer flow induced by a horizontal heated surface embedded in a porous medium. Hsu *et al.* (**13**) asserted that in this situation the longitudinal vortex is the favoured instability mode, but although they stated that this assumption follows from experimental evidence no reference to such work was given in their paper. The parallel-flow calculation predicts that longitudinal vortices grow beyond a non-dimensional distance of 33·47 from the leading edge. However, a similar analysis for wave disturbances by the present authors (**17**) shows that the corresponding critical distance is only 29·80. This suggests that waves constitute the more unstable of the two types of mode. This surprising result clearly means that one has to question the use of the parallel-flow approximation for this problem.

The result of the parallel-flow work described in (**13**) is that a unique stability curve is obtained for the longitudinal vortex modes. Sufficiently far downstream of the leading edge there are two vortex wavelengths (functions of the streamwise distance) that are neutrally stable. We shall see here that the parallel-flow results can only be relevant at asymptotically large distances from the leading edge. Furthermore, of the two 'neutral' modes predicted by the parallel theory one is indeed largely independent of the non-parallel terms and may be described in a manner akin to that in (**18**) for the Görtler problem; this accounts for the weak non-parallelism without formal difficulty. However, the second mode is sensitive to the non-parallel terms and we deduce that the concept of a unique neutral stability curve for this second vortex type is untenable. In addition, in the course of our investigation we are able to determine the most unstable mode, that is, the wavenumber and structure of the most rapidly growing longitudinal vortex.

The basic configuration we consider is shown in Fig. 1. A wedge-shaped region of angle α is composed of a fluid-saturated porous medium and is contained between two semi-infinite impermeable surfaces. One bounding surface ($y = 0, x \geqslant 0$) is inclined at an angle δ from the vertical whilst the other

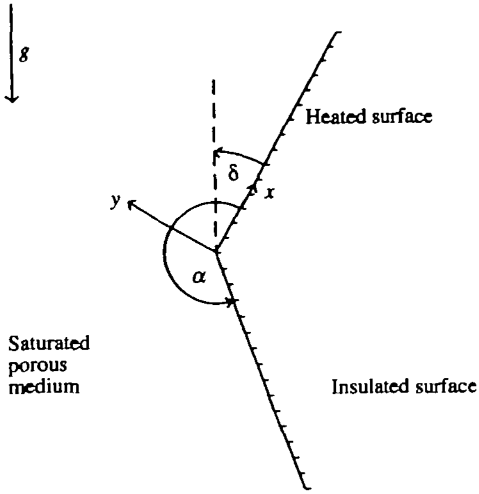

FIG. 1. The flow domain and coordinate system

is insulated; the ambient temperature of the medium is zero. All of our calculations will be confined to the case when $\delta = \frac{1}{2}\pi$, $\alpha = \frac{3}{2}\pi$ so that the heated surface is horizontal. Although this may appear to be a very specialized choice, the main attraction is that for this particular geometry we do have an exact solution for the underlying basic flow; see (**19**). Thus we can concentrate in this paper on assessing the effects of the non-parallelism of the vortices *without* resorting to an approximation of the basic flow. For other values of α and δ the boundary-layer flow can only be obtained approximately except for the one vertical case, $\delta = 0$, $\alpha = \pi$; see (**19**). We are currently investigating this more general problem which necessitates both approximating the underlying flow and describing non-parallel stability properties. It should be emphasized that our asymptotic description is only formally justifiable at large distances from the leading edge (this is completely analogous to the situation described by Hall (**20**) where the asymptotic analysis is valid only for large values of the Görtler parameter). In order to provide a complete description of the stability properties of the flow over the complete range of streamwise distances a numerical approach is required.

The procedure in the remainder of the paper is as follows. In section 2 we introduce the equations of motion and derive the linear stability equations for vortices in a boundary-layer flow above a horizontal surface. We briefly review the parallel analysis of the stability problem and then, in section 3, obtain the non-parallel solutions far downstream of the leading edge. This demonstrates how accounting for the effects of non-parallelism influences significantly the stability characteristics of the flow and, as a by-product, we are able to obtain the form of the most unstable vortex. In section 4 we close with a brief discussion and draw some conclusions.

2. The linear stability equations for longitudinal vortices

The basic configuration we consider is as described in the introduction and shown in Fig. 1. The bounding surface $y = 0$, $x \geqslant 0$ is horizontal whilst the second surface is vertical and insulated. We denote the fluid velocities in the x, y, z directions by u, v, w respectively, the pressure by p and the temperature by θ. Assuming that Darcy's law and the Boussinesq approximation are both valid the non-dimensional equations governing steady flow take the forms

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.1a}$$

$$u = -\frac{\partial p}{\partial x}, \qquad v = \theta - \frac{\partial p}{\partial y}, \qquad w = -\frac{\partial p}{\partial z}, \tag{2.1b, c, d}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}. \tag{2.1e}$$

Details of the non-dimensionalization involved may be found in (**21**). Since there is no natural length scale in the problem we have set the Darcy–Rayleigh number equal to unity; this defines a dimensional length scale in terms of the material parameters of the problem. In the absence of the vortex disturbance the basic flow is two-dimensional so that we write

$$(u, v) = \big(\bar{u}(x, y), \bar{v}(x, y)\big), \qquad p = \bar{p} \quad \text{and} \quad \theta = \bar{\theta}. \tag{2.2}$$

The continuity equation (2.1a) implies the existence of a streamfunction $\bar{\psi}$ such that $\bar{u} = -\bar{\psi}_y$, $\bar{v} = \bar{\psi}_x$ and elimination of the pressure between (2.1b, c) yields the governing equations for the basic steady flow:

$$\bar{\psi}_{xx} + \bar{\psi}_{yy} = -\bar{\theta}_x, \qquad \bar{\theta}_{xx} + \bar{\theta}_{yy} = \bar{\psi}_y\bar{\theta}_x - \bar{\psi}_x\bar{\theta}_y. \tag{2.3a, b}$$

In terms of polar coordinates, $x = r\cos\phi$, $y = r\sin\phi$, the boundary conditions may be written as

$$\bar{\psi} = 0, \quad \bar{\theta} = 1, \quad \text{on } \phi = 0, \tag{2.4a}$$

$$\bar{\psi} = 0, \quad \frac{\partial \bar{\theta}}{\partial \phi} = 0, \quad \text{on } \phi = \tfrac{3}{2}\pi, \tag{2.4b}$$

$$\bar{\theta} \to 0, \quad \bar{\psi} = o(r) \quad \text{as } r \to \infty, 0 < \phi < \tfrac{3}{2}\pi. \tag{2.4c}$$

We can solve for the basic flow by appealing to the exact solution given in (**19**). Equations (2.3a, b) subject to the boundary conditions (2.4a to c) are satisfied by

$$\bar{\psi} = \tfrac{1}{3}\xi\bar{f}(\eta), \qquad \bar{\theta} = \bar{g}(\eta), \tag{2.5a, b}$$

where the coordinates ξ and η are defined by $\xi = 3r^{\frac{1}{3}}\cos(\phi/3)$, $\eta = 3r^{\frac{1}{3}}\sin(\phi/3)$

or, alternatively, by

$$x = \frac{\xi}{27}(\xi^2 - 3\eta^2), \qquad y = \frac{\eta}{27}(3\xi^2 - \eta^2). \tag{2.6a, b}$$

The functions $\bar{f}$ and $\bar{g}$ satisfy

$$\bar{f}'' - \tfrac{2}{3}\eta\bar{g}' = 0, \qquad \bar{g}'' + \tfrac{1}{3}\bar{f}\bar{g}' = 0, \tag{2.7a, b}$$

subject to

$$\bar{f}(0) = 0, \quad \bar{g}(0) = 1, \qquad \bar{f}', \bar{g} \to 0 \quad \text{as } \eta \to \infty. \tag{2.7c}$$

Equations (2.7a, b) and boundary conditions (2.7c) were first obtained and solved numerically by Cheng and Chang (**22**) in their study of the leading-order boundary-layer flow far downstream of the leading edge.

2.1 *The disturbance equations*

We now perturb the basic flow (2.2), (2.3) in order to deduce the equations which govern linearized vortices within the boundary layer. We write

$$(u, v, w, p, \theta) = (\bar{u}, \bar{v}, 0, \bar{p}, \bar{\theta}) + \varepsilon\big(U(x, y, t), V(x, y, t), W(x, y, t), P(x, y, t), T(x, y, t)\big)E, \tag{2.8}$$

where we define $E \equiv \exp(iaz)$ and suppose that $\varepsilon \ll 1$. This form of disturbance allows us to investigate the properties of vortices with spanwise wavenumber a. On substituting (2.8) in (2.1), linearizing and then eliminating the disturbance velocities U, V and W from the five perturbation equations, we are left with the final coupled equations for the disturbance pressure P and temperature T:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - a^2 P = \frac{\partial T}{\partial y}, \tag{2.9a}$$

$$\frac{\partial T}{\partial t} + T\frac{\partial \bar{\theta}}{\partial y} + \bar{u}\frac{\partial T}{\partial x} + \bar{v}\frac{\partial T}{\partial y} - \frac{\partial P}{\partial x}\frac{\partial \bar{\theta}}{\partial x} - \frac{\partial P}{\partial y}\frac{\partial \bar{\theta}}{\partial y} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - a^2 T. \tag{2.9b}$$

In order to solve these perturbation equations we need to specify appropriate boundary conditions. From relations (2.6) and the basic flow solutions (2.7) it is clear that the thickness of the boundary layer which develops above the horizontal surface $y = 0$ is $O(x^{\frac{1}{3}})$. We shall be concerned with vortices within this layer so it is sufficient to state that, since the perturbation temperature and velocity normal to $y = 0$ must both vanish, we require that

$$T = \frac{\partial P}{\partial y} = 0 \quad \text{on } x > 0, \, y = 0.$$

We also demand that $T, P \to 0$ as we move out of the boundary layer. In addition, to solve (2.9) we must specify suitable initial conditions for the particular problem at hand. In the following analysis we shall be exploring the possible vortex structures for a prescribed vortex wavenumber a and shall not need to give detailed initial conditions.

Given the transformations (2.6) and the solution (2.7) for the basic flow it is straightforward to rewrite the disturbance equations (2.9) in the (ξ, η)-coordinate system;

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} - \frac{(\xi^2 + \eta^2)^2}{81} a^2 T = \frac{(\xi^2 - \eta^2)\bar{g}'}{9} T + \frac{\xi \bar{f}'}{3} \frac{\partial T}{\partial \xi} - \frac{\bar{f}}{3} \frac{\partial T}{\partial \eta} - \bar{g}' \frac{\partial P}{\partial \eta}, \quad (2.10a)$$

$$\frac{\partial^2 P}{\partial \xi^2} + \frac{\partial^2 P}{\partial \eta^2} - \frac{(\xi^2 + \eta^2)^2}{81} a^2 P = \frac{(\xi^2 - \eta^2)}{9} \frac{\partial T}{\partial \eta} + \frac{2\xi\eta}{9} \frac{\partial T}{\partial \xi}. \quad (2.10b)$$

It should be emphasized at this point that these disturbance equations are precise; for the particular configuration examined here no approximations have been invoked. A parallel-flow analysis of (2.10) involves specifying the form of the ξ-derivatives and thus reducing the governing partial differential equations to an ordinary differential form. We shall see how a formal asymptotic solution may be developed far downstream of the leading edge ($\xi \gg 1$) which allows the non-parallel terms to be taken into account without difficulty. We also note that in (2.10) we have taken the vortex to be stationary; thus we have suppressed the temporal-derivative term. This is because we shall be interested in examining the properties of the initial stages of the instability and, as in previous studies of longitudinal vortices (see (**8**, **9**)) and of Görtler vortices (**18**, **20**), we assume that in these initial stages the spatial evolution of the vortex structure is dominant over the temporal evolution.

Before moving to the non-parallel account of the problem it is useful to quickly summarize the parallel-flow results. This serves two purposes: first, it will enable us to see the importance of non-parallelism later and, secondly, it motivates many of the scalings in the following section.

Of importance in linear-stability studies is the neutral-stability curve which indicates the parameter combinations for which the vortex disturbance neither grows nor decays. A parallel-flow argument assumes that the ξ-dependence in the coefficients of the stability equations is suppressed so that we can effectively put $\partial/\partial\xi \equiv 0$ in (2.10b). If in addition it is assumed that the vortices are not 'wavy' then we are left with the linked equations

$$\frac{d^2 T}{d\eta^2} - \frac{(\xi^2 + \eta^2)^2}{81} a^2 T = \tfrac{1}{9}(\xi^2 - \eta^2)\bar{g}' T - \tfrac{1}{3}\bar{f} \frac{dT}{d\eta} - \bar{g}' \frac{dP}{d\eta}, \quad (2.11a)$$

$$\frac{d^2 P}{d\eta^2} - \frac{(\xi^2 + \eta^2)^2}{81} a^2 P = \tfrac{1}{9}(\xi^2 - \eta^2) \frac{dT}{d\eta}, \quad (2.11b)$$

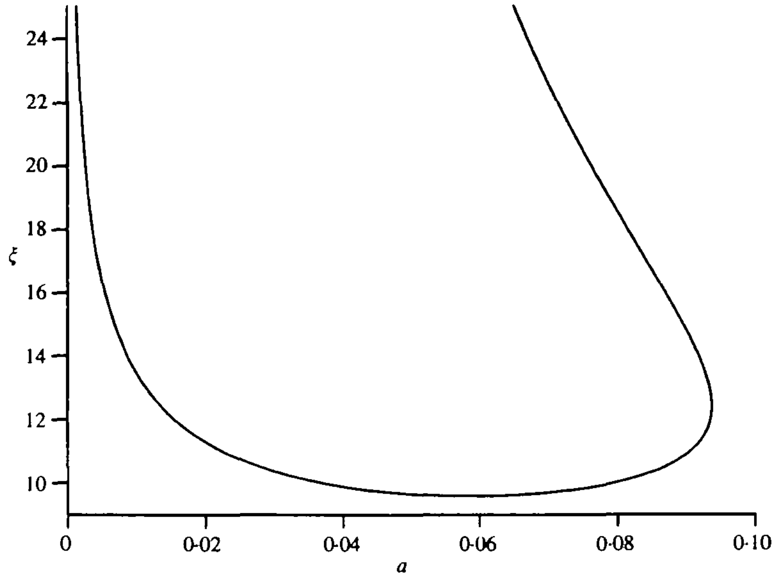

FIG. 2. The 'neutral curve' arising from solution of the parallel equations (2.11) subject to boundary conditions (2.12). Shown is the dependence of the vortex wavenumber a on the streamwise location ξ

subject to the boundary conditions

$$T = \frac{dP}{d\eta} = 0 \quad \text{on } \eta = 0; \qquad T, P \to 0 \quad \text{as } \eta \to \infty. \tag{2.12}$$

This constitutes an ordinary differential system in η and is an eigenvalue problem for the streamwise coordinate ξ in terms of the wavenumber a. A numerical solution of (2.11) is easily accomplished and results in the 'neutral stability curve' depicted in Fig. 2. For each downstream coordinate $\xi > \xi_c \approx 9{\cdot}7$ (or $x > \approx 33$ in agreement with the result of (**13**)) there exist two wavenumbers corresponding to neutral stability. The critical wavenumber is $a_c \approx 0{\cdot}059$ and, for wavenumbers $a > \approx 0{\cdot}094$ the vortices are predicted to be linearly stable at all distances downstream. Furthermore, it is clear from the numerical results that, as $\xi \to \infty$, the neutral wavenumbers approach zero; the 'left-hand' branch corresponds to $a = O(1/\xi^3)$ and the vortex structure then extends across the entire boundary layer $(\eta = O(1))$ whereas on the 'right-hand' branch $a = O(1/\xi)$ and the vortex is compressed into the thin region where $\eta = O(\xi^{-\frac{1}{2}})$. Details of these asymptotes will be postponed as they can be deduced as special cases of the non-parallel work to which we turn now.

3. Non-parallel analysis of induced vortices

Here we shall examine how non-parallelism can modify the conclusions of the simple parallel-flow analysis alluded to above. In common with other problems of this type we can only expect to be able to obtain an asymptotic description of the stability properties far downstream of the leading edge, $\xi \gg 1$. Clearly when $\xi = O(1)$ the ξ-derivatives in equations (2.10) are potentially as important as any of the other terms so that the stability characteristics of any particular disturbance at $O(1)$ downstream location are wholly dependent on how and where the disturbance commences. Thus we shall consider a position far downstream from the leading edge, say $\xi = \hat{\xi}$ ($\gg 1$). Let us investigate how a vortex evolves in the neighbourhood of $\hat{\xi}$ and concentrate first on modes near the right-hand branch of Fig. 2. From the results of the parallel analysis we suppose that such perturbations are confined to the region where $\eta = O(\hat{\xi}^{-\frac{1}{2}})$ and that the vortex wavenumber a is $O(\hat{\xi}^{-1})$. These choices reflect the need to provide a viscous balance and can be verified *a posteriori*. Furthermore, we shall not restrict ourselves to neutral modes alone and it is straightforward to show that for these modes the growth rate is $O(\hat{\xi})$. Finally, we need to consider the effects of non-parallelism, and the easiest method to deduce the appropriate scale is to demand that the non-parallel terms enter the governing equations at the stage at which the 'vertical' (that is, η-) structure of the mode is determined. This point will become clearer as the analysis proceeds but for the moment it suffices to say that the vortices evolve in a non-parallel manner within an $O(1)$ distance of $\hat{\xi}$.

Based on the discussion above we define the coordinate ζ by writing

$$\xi = \hat{\xi} + \zeta. \tag{3.1}$$

We anticipate that the disturbance is confined to the region where $\eta = O(\hat{\xi}^{-\frac{1}{2}})$ so we define the coordinate Y by

$$Y = \eta\hat{\xi}^{\frac{1}{2}} \tag{3.2a}$$

and seek solutions of (2.10) in the forms

$$a^2 = 81(A_0\hat{\xi}^{-2} + A_1\hat{\xi}^{-3} + \cdots), \tag{3.2b}$$

$$T = [T_0(\zeta, Y) + \hat{\xi}^{-1}T_1(\zeta, Y) + \cdots]\exp\left\{\hat{\xi}\int^{\zeta}\beta\, d\zeta\right\}, \tag{3.2c}$$

$$P = \hat{\xi}^{\frac{1}{2}}[P_0(\zeta, Y) + \hat{\xi}^{-1}P_1(\zeta, Y) + \cdots]\exp\left\{\hat{\xi}\int^{\zeta}\beta\, d\zeta\right\}, \tag{3.2d}$$

subject to the boundary conditions

$$T = \frac{dP}{dY} = 0 \quad \text{on } Y = 0, \qquad T, P \to 0 \quad \text{as } Y \to \infty. \tag{3.2e}$$

The parameter β gives the leading-order growth rate of the vortex and we shall determine how β varies with the wavenumber parameter A_0. It is worth mentioning at this point that in physical space the scaling $\eta = O(\hat{\xi}^{-\frac{1}{2}})$ suggests that at a distance $x \gg 1$ from the leading edge of the heated surface the vortex is concentrated in an $O(x^{\frac{1}{2}})$ depth region. We note that since the thermal boundary layer itself is of thickness $O(x^{\frac{1}{3}})$ the vortex is confined well inside this layer.

As we are concerned with small values of η we need to expand the basic flow functions $\bar{f}$, $\bar{g}$ about $\eta = 0$. Thus we write

$$\bar{f}(\eta) = f_{10}\eta + f_{30}\eta^3 + \dots, \qquad \bar{g}(\eta) = g_{10}\eta + g_{30}\eta^3 + \dots \quad (3.3\text{a, b})$$

for small η where, from (**19**), $g_{10} = -0{\cdot}43021$, $f_{10} = 1{\cdot}05575$ and we observe that $f_{30} = \frac{1}{9}g_{10}$ (<0) and $g_{30} = -f_{10}g_{10}/18$ (>0).

Substitution of (3.2) and (3.3) in (2.10a) yields at leading order

$$\beta^2 - \tfrac{1}{3}f_{10}\beta - (A_0 + \tfrac{1}{9}g_{10}) = 0, \tag{3.4}$$

which serves to give β as a function of the scaled wavenumber A_0. Equation (2.10b) gives that

$$(\beta^2 - A_0)P_0 = \frac{1}{9}\frac{\partial T_0}{\partial Y},$$

and thus P_0 is determined once we have T_0. We can find the latter by considering next-order terms in (2.10a) and so

$$\frac{\partial^2 T_0}{\partial Y^2} - \frac{(g_{10} + 3f_{10}\beta)(\beta f_{10} + \frac{1}{3}g_{30})}{(2g_{10} + 3f_{10}\beta)} Y^2 T_0 + \frac{(2\beta - \frac{1}{3}f_{10})(g_{10} + 3f_{10}\beta)}{(2g_{10} + 3f_{10}\beta)}\frac{\partial T_{10}}{\partial \zeta}$$
$$- \frac{(\frac{2}{9}g_{10} + \frac{1}{3}\beta f_{10} + 4A_0)(g_{10} + 3f_{10}\beta)}{(2g_{10} + 3f_{10}\beta)}\zeta T_0 - \frac{(g_{10} + 3f_{10}\beta)A_1}{(2g_{10} + 3f_{10}\beta)} T_0 = 0. \tag{3.5}$$

In view of the boundary conditions (3.2e) we seek solutions of (3.5) of the type $F(\zeta)G(Y)$ and such solutions are

$$T_0 = \exp\left\{\frac{[(\frac{2}{9}g_{10} + \frac{1}{3}\beta f_{10} + 4A_0)\zeta - \hat{\alpha}]^2}{2(2\beta - \frac{1}{3}f_{10})(\frac{2}{9}g_{10} + \frac{1}{3}\beta f_{10} + 4A_0)}\right\} U(-(2n + \tfrac{3}{2}), \sqrt{2}\Lambda^{\frac{1}{4}}Y), \tag{3.6a}$$

where

$$\Lambda \equiv \frac{(g_{10} + 3f_{10}\beta)(\beta f_{30} + \frac{1}{3}g_{30})}{(2g_{10} + 3f_{10}\beta)}, \tag{3.6b}$$

the constant $\hat{\alpha}$ is such that

$$\hat{\alpha} + A_1 = -\frac{(4n + 3)(2g_{10} + 3f_{10}\beta)}{(g_{10} + 3f_{10}\beta)}, \qquad n = 0, 1, 2, \dots \tag{3.6c}$$

and U is the usual parabolic-cylinder function, see (**23**). However, we do need to impose the restriction on Λ that it should not be negative for in this case the parabolic-cylinder function defined in (3.6a) does not have acceptable exponential decay as $Y \to \infty$. Further, we note that relation (3.6c) relates the constant $\hat{\alpha}$ in the ζ-part of solution (3.6a) to the correction term A_1 in the vortex wavenumber expansion (3.2b).

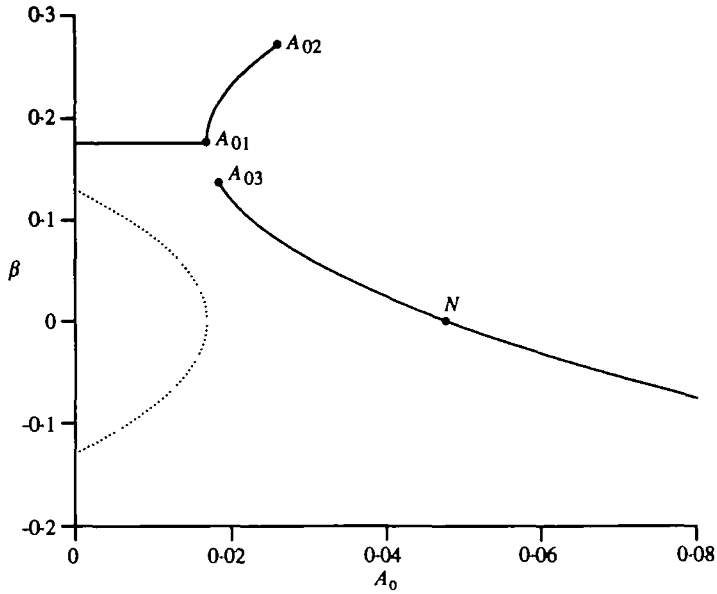

FIG. 3. The scaled vortex growth rate β as a function of the wavenumber parameter A_0 (defined in (3.2b)). The solid lines denote Re(β) and the dotted lines Im(β). (The dotted line is only shown for wavenumbers at which Im(β) $\neq$ 0.) The points labelled A_{01}, A_{02}, A_{03} and N are referred to in the text. The last of these denotes the one neutral mode within this wavenumber regime

We are now in a position to find β as a function of A_0. We simply solve the quadratic equation (3.4) for β as a function of A_0 and then ensure that Λ is not negative. The acceptable solutions of this problem are shown in Fig. 3. Briefly, we can summarize the results as follows. For $0 < A_0 < A_{01}$ ($\approx 0{\cdot}01683$) the two roots for β are complex and the growth rate of the modes Re(β) is $\frac{1}{6}f_{10} = 0{\cdot}17596$. For A_0 just greater than A_{01} the two values of β are purely real and the growth-rate curve develops two branches. The upper of these increases with A_0 until $A_0 = A_{02} = 0{\cdot}02598$ at which point $\beta = 0{\cdot}27161$. As $A_0 \to A_{02}$ the coefficient Λ becomes infinite and so solution (3.6a) indicates that the vortex structure is compressed into a very thin layer in which $Y = O(\Lambda^{-\frac{1}{4}})$ next to the wall. A consequence of this is that when $A_0 > A_{02}$ the upper branch cuts out, for Λ becomes negative and so this particular solution of (3.6) ceases to be acceptable. Meanwhile, on the lower branch we do not obtain permissible structures until A_0 reaches $A_{03} = 0{\cdot}01844$ —thereafter on this branch the growth rate diminishes with A_0 and passes through a neutral point at $A_0 = 0{\cdot}04780$ (and is denoted by N on Fig. 3).

On the basis of the evidence so far we would conclude that the most unstable mode occurs at $A_0 = 0{\cdot}02598$ (or $a \sim 1{\cdot}45/\xi + \cdots$) and we should consider whether there is a faster growing mode elsewhere. In passing we should note the role that non-parallelism plays in determining the structure of the modes in the vicinity of the right-hand branch of the parallel-theory curve in Fig. 2. A parallel analysis follows the same lines as described above except that all ζ-dependence in $T_0, P_0, T_1, P_1, \ldots$ is ignored. Then the computed neutral point is precisely that prediced by our work and so the effects of non-parallelism are not significant here. However, solution (3.6a) shows that the (non-parallel) exponential part of the vortex solution will grow or decay as it progresses downstream depending on whether $(2\beta - \frac{1}{3}f_{10})(\frac{2}{9}g_{10} + \frac{1}{3}\beta f_{10} + 4A_0)$ is positive or negative. (It should be emphasized that the dominant change to the growth rate as a result of downstream evolution arises parametrically through (3.1), (3.2b) and (3.4).) As would be expected in view of Fig. 3, growth of the exponential term is found to occur for modes on the upper branch whilst for lower-branch disturbances the exponential factor decays.

3.1 *The behaviour of the upper branch mode as $A_0 \to A_{02}$*

It is of importance to determine the fate of the upper-branch mode as $A_0 \to A_{02}$ and it is not difficult to examine the structure of the vortex in this limit. In the process of doing this we show that another family of vortex modes exist which are characterized by being confined to an $O(1)$ depth region in terms of the η coordinate. Let us choose $\eta = \bar{\eta}$ and in an $O(\xi^{-\frac{2}{3}})$ neighbourhood of $\bar{\eta}$ we define the coordinate $\tilde{Y}$ by

$$\eta = \bar{\eta} + \xi^{-\frac{2}{3}}\tilde{Y}, \tag{3.7a}$$

and seek modes with

$$a^2 = 81(\tilde{A}_0\xi^{-2} + \hat{A}_1\xi^{-\frac{8}{3}} + \cdots), \tag{3.7b}$$

$$T = [\tilde{T}_0(\zeta, \tilde{Y}) + \xi^{-\frac{2}{3}}\tilde{T}_1(\zeta, \tilde{Y}) + \cdots] \exp\left\{\hat{\xi} \int^{\zeta} (\tilde{\beta}_0 + \xi^{-\frac{2}{3}}\tilde{\beta}_1 + \cdots)\, d\zeta\right\}, \tag{3.7c}$$

$$P = [\tilde{P}_0(\zeta, \tilde{Y}) + \xi^{-\frac{2}{3}}\tilde{P}_1(\zeta, \tilde{Y}) + \cdots] \exp\left\{\hat{\xi} \int^{\zeta} (\tilde{\beta}_0 + \xi^{-\frac{2}{3}}\tilde{\beta}_1 + \cdots)\, d\zeta\right\}. \tag{3.7d}$$

If in the vicinity of $\bar{\eta}$ the basic flow quantities $\bar{f}$, $\bar{g}$ expand as

$$\left.\begin{aligned} \bar{f} &= f_0 + f_1\xi^{-\frac{2}{3}}\tilde{Y} + f_2\xi^{-\frac{4}{3}}\tilde{Y}^2 + \ldots, \\ \bar{g} &= g_0 + g_1\xi^{-\frac{2}{3}}\tilde{Y} + g_2\xi^{-\frac{4}{3}}\tilde{Y}^2 + \ldots, \end{aligned}\right\} \tag{3.7e}$$

then equations (2.10) give that

$$\tilde{\beta}_0^2 - \tilde{A}_0 = \tfrac{1}{9}g_1 + \tfrac{1}{3}f_1\tilde{\beta}_0, \tag{3.8a}$$

$$\frac{\partial^2 \tilde{T}_0}{\partial \tilde{Y}^2} - \frac{(g_1 + 3\tilde{\beta}_0 f_1)(\frac{2}{9}g_2 + \frac{2}{3}\tilde{\beta}_0 f_2)}{(2g_1 + 3\tilde{\beta}_0 f_1)}\tilde{Y}\tilde{T}_0 + \frac{(g_1 + 3\tilde{\beta}_0 f_1)(6 - f_1)}{3(2g_1 + 3\tilde{\beta}_0 f_1)}\tilde{\beta}_1\tilde{T}_0 = 0. \tag{3.8b}$$

This latter equation is a scaled Airy equation and in order to confine the disturbance to lie below $\eta = \bar{\eta}$ we should demand that $\tilde{T}_0$ decays exponentially as $\tilde{Y} \to \infty$ and this is possible if

$$\tilde{\Lambda} \equiv \frac{(g_1 + 3\tilde{\beta}_0 f_1)(\frac{2}{9}g_2 + \frac{2}{3}\tilde{\beta}_0 f_2)}{(2g_1 + 3\tilde{\beta}_0 f_1)} > 0. \tag{3.9}$$

If

$$\tilde{\tilde{Y}} \equiv \tilde{\Lambda}^{\frac{1}{3}}\left(\tilde{Y} - \frac{3(6 - f_1)}{2(g_2 + 3\tilde{\beta}_0 f_2)}\right)$$

then

$$\frac{\partial^2 \tilde{T}_0}{\partial \tilde{\tilde{Y}}^2} - \tilde{\tilde{Y}}\tilde{T}_0 = 0$$

and $\tilde{T}_0 \sim (-\tilde{\tilde{Y}})^{-\frac{1}{4}} \sin(\frac{2}{3}(-\tilde{\tilde{Y}})^{\frac{3}{2}} + \frac{1}{4}\pi)$ as $\tilde{\tilde{Y}} \to -\infty$; see (**23**). Thus in the region $0 < \eta < \bar{\eta}$ below the $O(\xi^{-\frac{2}{3}})$ depth layer we see that for matching we must ensure that

$$T \sim \xi^{-\frac{1}{6}}(\bar{\eta} - \eta)^{-\frac{1}{4}} \sin(\tfrac{2}{3}\Lambda^{\frac{1}{2}}\xi(\bar{\eta} - \eta)^{\frac{3}{2}} + \cdots) \quad \text{as } \eta \to \bar{\eta}. \tag{3.10}$$

A standard WKB analysis may be used to develop the forms of the disturbance pressure P and temperature T in $0 < \eta < \bar{\eta}$ so that the requisite boundary condtions $T = dP/d\eta = 0$ are achieved on the bounding surface and that a match is obtained with (3.10). Details of this WKB process are long but straightforward and we shall not present them here. The primary reason for this can be seen from the following argument. If this second type of mode (as distinct from the wall-bounded type discussed previously) is to exist then we need to solve (3.8*a*) for the growth rate $\tilde{\beta}_0$ and then check that the condition (3.9) is satisfied at $\eta = \bar{\eta}$. Equations (2.7) for the basic flow show that, for all η, $\bar{f}'$ is monotonically decreasing and $\bar{g}'$ is monotonically increasing. It transpires that for any prescribed wavenumber parameter A_0 the greatest root for $\tilde{\beta}_0$ occurs when $\bar{\eta} = 0$. In this case the mode is trapped at the wall and is essentially the first vortex type discussed in (3.1) to (3.6) and plotted on Fig. 3. Thus the wall-bounded modes are more unstable than those confined to the $\eta = O(1)$ zone. Furthermore, for each $\tilde{A}_0$ there is a value $\bar{\eta}_{\text{crit}}(\tilde{A}_0)$ such that the second type of mode can lie in the zone $0 < \eta < \bar{\eta}$ where $\bar{\eta} < \bar{\eta}_{\text{crit}}$. It is found that as $\tilde{A}_0 \to A_{02}$ (recall that this is the value at which the upper branch on Fig. 3 cuts out) then $\bar{\eta}_{\text{crit}} \to 0$ and the two distinct mode types coalesce. Then for $\tilde{A}_0 > A_{02}$ neither type of mode can persist near the upper branch and this family of solutions disappears.

This brief explanation of the fate of the upper branch also holds for the lower branch as $A_0 \to A_{03}^{+} = 0{\cdot}01844$. At this vortex wavenumber the lower-branch modes of Fig. 3 coalesce with those confined to an $\eta = O(1)$ region and neither type remains when $A_0 < A_{03}$. Consequently, we can conclude that within the $a = O(1/\xi)$ range unstable modes exist for $a < 1{\cdot}97/\xi$. Furthermore, the most unstable mode has $a = 1{\cdot}45/\xi$ and scaled growth rate $\beta = 0{\cdot}27161$. For all $a > 1{\cdot}97/\xi$ the modes are stable and thus decay as they progress downstream.

We have seen that at vortex wavenumber $a = O(1/\hat{\xi})$ non-parallel effects are not critical in determining the stability characteristics of the vortex modes. However, the inclusion of non-parallelism has enabled us to deduce how the modes evolve far downstream; see (3.6) for example. In practice, if an infinitesimal vortex of wavenumber $O(1/\hat{\xi})$ is introduced some distance downstream of the leading edge then it will be composed of a linear combination of the various types of modes described above and it will be the most unstable component of this combination which will ultimately dominate the flow. Except in very special circumstances the initial vortex will contain wall-bounded components and the growth of these were given in Fig. 3.

3.2 *The fully non-parallel mode*

We now examine the vortices in the vicinity of the left-hand branch of the parallel-flow neutral curve sketched in Fig. 2 and we shall see how non-parallelism plays a much more important part here. We have already asserted that in this regime we have

$$a^2 = 81\left(\frac{\hat{A}_0}{\hat{\xi}^6} + \frac{\hat{A}_1}{\hat{\xi}^8} + \cdots\right) \tag{3.11}$$

and that the vortex lies across the whole of the $\eta = O(1)$ thermal boundary layer. As in our preceding analysis of the $O(1/\hat{\xi})$ wavenumber modes we allow the vortices to develop on the lengthscale such that non-parallel terms enter the determining equations at the point at which the η-structure of the mode is deduced. This implies that the vortex evolves on a long, $O(\xi)$, lengthscale and so we define the $O(1)$ coordinate $\hat{\zeta}$ by

$$\hat{\zeta} = \xi/\hat{\xi}, \tag{3.12a}$$

and seek modes which are initiated at $\xi = \hat{\xi}$; that is, at $\hat{\zeta} = 1$. We then study perturbations of the forms

$$P = \hat{P}_0(\hat{\zeta}, \eta) + \hat{\xi}^{-2}\hat{P}_1(\hat{\zeta}, \eta) + \ldots, \tag{3.12b}$$

$$T = \hat{\xi}^{-2}\hat{T}_0(\hat{\zeta}, \eta) + \hat{\xi}^{-4}\hat{T}_1(\hat{\zeta}, \eta) + \ldots. \tag{3.12c}$$

Substitution of (3.11), (3.12) in (2.10) gives at leading orders a first relation between $\hat{P}_0$ and $\hat{T}_0$:

$$\frac{\partial \hat{P}_0}{\partial \eta} = \tfrac{1}{9}\hat{\zeta}^2 \hat{T}_0. \tag{3.13a}$$

At next orders we have

$$\frac{\partial^2 \hat{T}_0}{\partial \eta^2} = \frac{1}{9}\bar{g}'[\hat{\zeta}^2\hat{T}_1 - \eta^2\hat{T}_0] + \frac{1}{3}\hat{\zeta}\bar{f}'\frac{\partial \hat{T}_0}{\partial \hat{\zeta}} - \frac{1}{3}\bar{f}\frac{\partial \hat{T}_0}{\partial \eta} - \bar{g}'\frac{\partial \hat{P}_1}{\partial \eta} \tag{3.13b}$$

and

$$\frac{\partial^2 \hat{P}_1}{\partial \eta^2} - \frac{1}{9}\hat{\zeta}^2 \frac{\partial \hat{T}_1}{\partial \eta} = \hat{A}_0 \hat{\zeta}^4 \hat{P}_0 + \frac{2}{9}\eta\hat{\zeta}\frac{\partial \hat{T}_0}{\partial \hat{\zeta}} - \frac{1}{9}\eta^2 \frac{\partial \hat{T}_0}{\partial \eta} - \frac{\partial^2 \hat{P}_0}{\partial \hat{\zeta}^2}, \qquad (3.13c)$$

so that on eliminating $\hat{P}_1$ and $\hat{T}_1$ we obtain the evolution equation for the leading-order perturbation pressure $\hat{P}_0$:

$$9\frac{\partial^4 \hat{P}_0}{\partial \eta^4} + 6\bar{f}\frac{\partial^3 \hat{P}_0}{\partial \eta^3} + (9\bar{f}' + \bar{f}^2)\frac{\partial^2 \hat{P}_0}{\partial \eta^2} + 2(\eta\bar{g}' + \bar{f}\bar{f}')\frac{\partial \hat{P}_0}{\partial \eta} + \hat{A}_0 \bar{g}'\hat{\zeta}^6 \hat{P}_0$$
$$= 3\hat{\zeta}\bar{f}'\frac{\partial^3 \hat{P}_0}{\partial \eta^2 \partial \hat{\zeta}} + \bar{f}\bar{f}'\hat{\zeta}\frac{\partial^2 \hat{P}_0}{\partial \eta \partial \hat{\zeta}} + \bar{g}'\hat{\zeta}^2 \frac{\partial^2 \hat{P}_0}{\partial \hat{\zeta}^2}. \qquad (3.14)$$

In previous analyses it has been argued that since the disturbance develops over a long lengthscale the streamwise derivatives on the right-hand side of (3.14) should be small and hence may safely be neglected. Then, if we seek modes which are neutral at $\xi = \hat{\xi}$ the argument pursued in (**8**, **9**, **12**, **13**) is that we can suppose that $\zeta = 1$ and the $\partial/\partial\hat{\zeta}$ terms in (3.14) are put to zero. In this way we can obtain an ordinary differential equation in η which needs to be solved subject to

$$\frac{d\hat{P}_0}{d\eta} = \frac{d^3 \hat{P}_0}{d\eta^3} = 0 \quad \text{at } \eta = 0; \qquad \hat{P}_0 \to 0 \quad \text{as } \eta \to \infty,$$

which is an eigenproblem for the leading-order wavenumber $\hat{A}_0$. However, there is of course no rational argument why the $\hat{\zeta}$ derivatives should vanish and the full version of (3.14) must be considered in order to evaluate the stability characteristics of the vortex modes. Pursuing the analogy with the Görtler problem considered by Hall (see (**24**)) it is clear that the stability properties of the flow are strongly dependent on the position and form of the initial vortex. We have not undertaken a numerical solution of (3.14) as we can see that the modes of wavenumber $a = O(1/\hat{\xi})$ are those with the larger growth rates and therefore are almost certainly the more important in practice.

It might be hoped that by taking suitable limits in the two problems it may be possible to connect the scalings appropriate to the $a = O(1/\hat{\zeta})$ and $a = O(1/\hat{\xi}^3)$ regimes. However, there is no direct link between these cases which indicates the existence of at least one more distinct wavenumber regime between the two discussed here. We argue that a detailed analysis of such an intermediate problem is not of primary importance for it is easy to show that such modes have growth rates less than those associated with the $a = O(1/\hat{\xi})$ region. Therefore, we can be confident that it is this latter regime which contains the most unstable form.

Our workings of this section can be used to speculate how vortices might evolve in the boundary layer above a heated horizontal surface. Before we outline this it has to be emphasized that problem (2.10) is inherently elliptic and not parabolic in nature. Therefore, we cannot just allow a vortex structure to be imposed at some downstream location and suppose that it will evolve for downstream effects can, and will, act back on the mode and thus influence its development. However, bearing this caution in mind, we can appeal to experimental observations that as the vortex progresses in the flow its physical wavelength is seen to be preserved. Since the basic boundary layer thickens with distance the effective local spanwise wavenumber diminishes and, as a consequence, the small a analysis is of practical interest. As a vortex progresses downstream our analysis suggests that the perturbation decays until the $\xi = O(a^{-\frac{1}{3}})$ non-parallel scaling is encountered. At some point within this $\xi = O(a^{-\frac{1}{3}})$ region the 'left-hand' branch of the neutral curve is crossed and the disturbance begins to grow. Recall that the precise location within the $O(a^{-\frac{1}{3}})$ regime at which vortex growth may commence is entirely dependent on how and where the vortex motion started as (3.14) is controlled by non-parallel effects. As the flow develops further downstream the vortex amplitude is likely to increase until the $\xi = O(a^{-1})$ quasi-parallel region is reached. We have shown that within this region we have large growth rates for the disturbance (cf. (3.2)). Although the linear stability analysis of modes within this regime is complicated it is very likely that the massive growth rates will mean that a full account of the vortex motion will require a nonlinear or, preferably, a computational description. However, on a purely linear basis, the analysis of this section would suggest that the vortex moves through the $\xi = O(a^{-1})$ scaling region so that the amplitude of the perturbation is likely to grow to a maximum and then lead to eventual decay. Thus solutions of (2.10) can be anticipated to decay initially until the amplitude reaches a minimum and then continue growing further downstream until it reaches a maximum after which the mode shrinks to zero.

4. Discussion

In this paper we have concentrated on describing some of the linear stability properties of longitudinal vortices in the boundary layer above a horizontal heated surface in a porous medium. We have successfuly obtained an asymptotic description of modes far downstream of the leading edge of the plate using methods similar to those employed by Hall (**18**, **20**, **24**) for Görtler vortices. Previous investigation of vortices in flow over porous surfaces have largely utilized parallel-flow analyses which have concluded that at any specified location sufficiently far downstream there are two vortex wavelengths that are neutrally stable. Examination of the role of non-parallelism has shown that of the two neutral modes predicted by the parallel work, the one with the shorter wavelength can be described by a quasi-parallel analysis whilst the other is dominated by non-parallel effects.

Our analysis of vortex modes has enabled us to deduce the structure of the most unstable disturbance far downstream of the leading edge. This aspect has been discussed in the preceding section where we also observed that the form of the wavenumber/growth-rate curve is quite complicated (cf. the simple form of this dependence for Görtler vortices; see (**25**)). The corresponding Görtler problem is comparatively simple due to the fact that the underlying governing equations are then parabolic in nature: here we are dealing with the elliptic equations (2.10). This then leads to a linear as opposed to our quadratic equation for the leading-order growth rate and this significantly simplifies the ensuing analysis.

The asymptotic work we have presented here needs to be complemented by suitable numerical studies. We note that since our present results are only strictly valid at large downstream distances we need numerical investigations in order to deduce the stability properties at locations close to the leading edge. One way out of this difficulty is to employ an elliptic non-parallel theory for the linearized disturbances, that is, a fully numerical simulation of equations (2.10). Such an analysis has the great advantage that the disturbances can be allowed to be self-propagating. The non-parallel analysis devised by Hall (**24**) for the Görtler case is parabolic and, therefore, it requires a disturbance to be imposed at a given streamwise location for all time. An elliptic non-parallel analysis is an initial-value problem; a disturbance is defined at one instant in time and is allowed to evolve. Work on this approach is in progress.

Numerical work is also required in order to describe correctly the nonlinear evolution of the vortices examined here. In our discussion of the evolution of infinitesimal vortices we speculated that it might be plausible that as such vortices propagate downstream they initially decay until the non-parallel scaling is reached, then grow until the quasi-parallel regime is achieved and finally decay away. However, the complicated nature of the $\xi = O(a^{-1})$ quasi-parallel regime that numerical procedures really do need to be implemented in order to confirm that this is the actual sequence of events. Within the growing stage it is quite conceivable that the linear theory ceases to remain valid and nonlinear effects need to be accounted for. In many fluid-flow situations nonlinearity may be described by a classical Stuart–Watson theory as detailed in (**26** to **28**) but for Görtler vortex flows Hall (**18**) demonstrated that the nonlinear evolution of high-wavenumber Görtler modes is dictated by a mean-field theory. In this case the downstream velocity component of the perturbation contains a mean-flow correction which is as large as the fundamental driving it. These functions satisfy a pair of nonlinear partial differential equations which must be solved subject to some initial conditions imposed at some given downstream location. Preliminary investigations of the convective problem examined in this paper suggests that it too is governed by a mean-field approach akin to that described in (**18**); we hope to be able to report on this in due course.

Finally, we comment on how our work could be extended to other problems concerning vortex modes in porous media. In our current study one reason for our concentrating on the horizontal heated surface case, apart from its practical interest for geothermal systems, is that an exact basic-flow solution exists. This enabled us to isolate the nonparallel properties of the flow. However, when the surface is arbitrarily inclined then the basic flow is obtained only in the form of an asymptotic series in terms of the (large) distance from the leading edge so that it is more difficult to differentiate between the effects of nonparallelism on the disturbance and those of the spatial development of the basic flow. It is thought that when the heated surface is vertical then the flow is stable to vortex modes of all wavelengths so that one would expect that as the heated surface is incremented from horizontal to vertical there is a critical inclination at which the stability characteristics of the flow must change substantially from those presented in this study. Thus we have given here the first non-parallel investigation of vortices in porous media which provides the framework for extension to other configurations and nonlinear stability studies. Complementing numerical and theoretical studies of travelling-wave modes should enable us to provide the definitive account of the relative importance of wave and vortex disturbances in a variety of practical boundary-layer flows in porous media.

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