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# The Effect of Local Thermal Nonequilibrium on the Stability of Convection in a Vertical Porous Channel

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**Abstract** We consider the effect of local thermal nonequilibrium on the stability properties of convection in a vertical porous channel heated and cooled from the sides. On using an energy stability analysis of the linearised stability equations, we show that the system remains unconditionally stable to small-amplitude disturbances.

**Keywords** Vertical porous layer · Stability analysis · Local thermal nonequilibrium

## List of Symbols

| | |
|---|---|
| $c$ | Specific heat |
| $g$ | Gravity |
| $h$ | Interfacial heat transfer coefficient |
| $H$ | Nondimensional form of $h$ |
| $k$ | Disturbance wavenumber |
| $L$ | Width of the layer |
| $K$ | Permeability |
| $p$ | Pressure |
| $Ra$ | Darcy–Rayleigh number |
| $t$ | Time |
| $u$ | Horizontal velocity |
| $v$ | Vertical velocity |
| $x$ | Horizontal coordinate |
| $y$ | Vertical coordinate |

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**Greek Symbols**

$\alpha$ Diffusivity ratio
$\beta$ Thermal expansion coefficient
$\gamma$ Scaled conductivity ratio
$\theta$ Fluid temperature
$\Theta$ Disturbance fluid temperature
$\overline{\Theta}$ Complex conjugate of $\Theta$
$\lambda$ Exponential growth rate
$\mu$ Dynamic viscosity
$\rho$ Density
$\phi$ Solid temperature
$\Phi$ Disturbance solid temperature
$\overline{\Phi}$ Complex conjugate of $\Phi$
$\psi$ Stream function
$\Psi$ Disturbance stream function

**Subscripts and Superscripts**

c Cold surface
f Fluid phase
h Hot surface
s Solid phase
ref Reference value
$'$ Derivative with respect to $x$
$\hat{}$ Dimensional

## 1 Introduction

In this short article, we adopt the method used by Gill (1969) in order to study the effect of local thermal nonequilibrium on the stability properties of convection in a vertical porous layer for which one boundary is uniformly cold and the other is uniformly hot. In what is perhaps the shortest ever paper in the Journal of Fluid Mechanics, Gill used an energy analysis of the linearised disturbance equations to show that the basic state, which consists of linearly varying temperature and vertical velocity fields, is linearly stable.

There are very few works which consider the stability of convection in vertical channels. Rees (1988) considered the same problem as Gill but included the Prandtl–Darcy term (a time derivative of the velocity) in the momentum equations. Numerical solutions were presented which show that the decrement spectrum (i.e. the variation of the real part of the exponential growth rate with Darcy–Rayleigh number, $Ra$) is modified substantially by the presence of this term, but that convection remains stable. Later work by Lewis et al. (1995) quantified Gill's results by determining how quickly disturbances decay; they showed that the exponential decay rate of disturbances is proportional to $(k\,Ra)^{2/3}$ at leading order when $Ra$ is asymptotically large and where $k$ is the wavenumber of the disturbances. Thus, the layer is increasingly stable to two-dimensional disturbances as $Ra$ increases.

Kwok and Chen (1987) carried out experiments using a porous medium composed of glass beads and distilled water and found situations in which convective instability arises. They undertook two linear stability analyses for the case where Darcy's law is supplemented by both the Brinkman terms and the advective inertia terms. In one analysis, they used a constant viscosity while in the other they used the variation of viscosity with temperature which corresponded to their experiment. In each case, they obtained a neutral stability curve, but the onset criterion for the former analysis corresponded to a physically unrealistic temperature difference across the layer (although a wider layer would reduce the required temperature difference to realistic values). However, neither critical Darcy–Rayleigh number corresponded to the one obtained experimentally.

Kwok and Chen (1987) also discussed the different qualitative behaviours of the Darcy-flow model used by Gill (1969) and the full Darcy–Brinkman model with advective inertia. They concluded that the no-slip condition has a 'minimal influence on the stability' even though they presented a neutral curve for a constant viscosity fluid. Gill (1969) compared his result with that for a clear fluid and concluded that instability does not arise for Darcy flow because of the absence of advective inertia. At present, however, the separate roles of advective inertia and the Brinkman terms remain unknown, and therefore, an analysis with the Brinkman terms but without the advective inertia terms would be of great interest.

Further papers on this topic include one by Straughan (1988) and the later work of Qin and Kaloni (1993). Both papers use energy methods to give sufficient conditions for stability. In particular, Straughan (1988) showed that Gill's conclusion also applies to all nonlinear disturbances.

In this paper, we revisit the analysis of Gill (1969) and consider the effect of local thermal equilibrium between the phases. Briefly, this is the situation where the temperatures of the solid and fluid phases have to be described using different heat transport equations, and therefore, the respective temperatures at a chosen point might indeed be different. That this can be so arises because these are macroscopic temperatures which are averages over small representative elementary volumes. In previous work, it has been found that the presence of local thermal nonequilibrium can alter substantially the stability properties of the Darcy–Bénard problem, namely a horizontal porous layer heated from below. Very strong variations in the critical Darcy–Rayleigh number and wavenumber were found; see Banu and Rees (2002). Therefore, it is of interest to determine whether any qualitative change takes place for the vertical layer from what was found by Gill (1969).

## 2 Governing Equations

We consider the stability of convection in a vertical porous layer of width, $L$, lying in the region $-L/2 \leq \hat{x} \leq L/2$, and which is of infinite extent vertically, i.e. in the $y$-direction. Darcy's law and the Boussinesq approximation both apply, but the phases are not in local thermal equilibrium. Therefore, the full two-dimensional governing equations are given by,

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0, \tag{1a}$$

$$\hat{u} = -\frac{K}{\mu}\frac{\partial \hat{p}}{\partial \hat{y}}, \quad \hat{v} = -\frac{K}{\mu}\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\rho_{\mathrm{f}} g \beta K}{\mu}\left(T_{\mathrm{f}} - T_{\mathrm{ref}}\right), \tag{1b,c}$$

$$\epsilon(\rho c)_{\mathrm{f}}\frac{\partial T_{\mathrm{f}}}{\partial \hat{t}} + (\rho c)_{\mathrm{f}}\left(u\frac{\partial T_{\mathrm{f}}}{\partial \hat{x}} + v\frac{\partial T_{\mathrm{f}}}{\partial \hat{y}}\right) = \epsilon k_{\mathrm{f}}\nabla^2 T_{\mathrm{f}} + h(T_{\mathrm{s}} - T_{\mathrm{f}}), \tag{1d}$$

$$(1-\epsilon)(\rho c)_s\frac{\partial T_s}{\partial \hat{t}} = (1-\epsilon)k_{\mathrm{s}}\nabla^2 T_{\mathrm{s}} - h(T_{\mathrm{s}} - T_{\mathrm{f}}); \tag{1e}$$

see Nield and Bejan (2006). The quantity, $h$, is the interfacial heat transfer coefficient which depends on the physical properies of the two phases, the geometry of the porous structure, and the local fluid velocity. The boundary conditions are that,

$$\hat{u} = 0, \quad T_{\rm f} = T_{\rm s} = T_{\rm c} \quad \text{on} \quad \hat{x} = -L/2, \quad \hat{u} = 0, \quad T_{\rm f} = T_{\rm s} = T_{\rm h} \quad \text{on} \quad \hat{x} = L/2, \tag{2}$$

where $T_{\rm c} < T_{\rm h}$. The reference temperature which appears in the buoyancy term in Eq. 1 is given by

$$T_{\rm ref} = (T_{\rm c} + T_{\rm h})/2. \tag{3}$$

Equations 1 may be nondimensionalised using the transformations,

$$(\hat{x}, \hat{y}) = L(x, y), \quad (\hat{u}, \hat{v}) = \frac{\epsilon k_{\rm f}}{(\rho c)_{\rm f} L}(u, v), \quad \hat{p} = \frac{k_{\rm f}\mu}{(\rho c)_{\rm f} K} p, \tag{4a,b,c}$$

$$T_{\rm f} = (T_{\rm h} - T_{\rm c})\theta + T_{\rm ref}, \quad T_{\rm s} = (T_{\rm h} - T_{\rm c})\phi + T_{\rm ref}, \tag{4d,e}$$

$$\bar{t} = \frac{(\rho c)_{\rm f}}{k_{\rm f}} L^2 t. \tag{4f}$$

For two-dimensional flow, we may define the streamfunction, $\psi$, according to $u = -\psi_y$ and $v = \psi_x$ and Eqs. 1 now become

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = Ra\, \frac{\partial \theta}{\partial x}, \tag{5a}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + H(\phi - \theta), \tag{5b}$$

$$\alpha \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \gamma H(\theta - \phi), \tag{5c}$$

where

$$Ra = \frac{\rho_{\rm f} g \beta (T_{\rm h} - T_{\rm c}) K L}{\epsilon \mu \kappa_{\rm f}}, \quad \gamma = \frac{\epsilon k_{\rm f}}{(1 - \epsilon) k_{\rm s}}, \tag{6a,b}$$

$$H = \frac{h L^2}{\epsilon k_{\rm f}}, \quad \text{and} \quad \alpha = \frac{(\rho c)_{\rm s}}{(\rho c)_{\rm f}} \frac{k_{\rm f}}{k_{\rm s}}, \tag{6c,d}$$

are the Darcy–Rayleigh number based on the fluid properties, a porosity-modified conductivity ratio, a scaled inter-phase heat transfer coefficient, and a diffusivity ratio, respectively. The boundary conditions become,

$$\psi = 0, \quad \theta = \phi = -1/2 \quad \text{on} \quad x = -1/2, \quad \psi = 0, \quad \theta = \phi = 1/2 \quad \text{on} \quad x = 1/2. \tag{7}$$

We note that Rees and Bassom (2000), who considered stability in an inclined channel, showed that Squire's theorem applies for their system which includes a vertical channel as a special case, and therefore, it is a small matter to show that it also applies here. Therefore, we may confine our analysis to two dimensions without loss of generality.

## 3 Linear Stability Analysis

Equations 4 admit the following basic solution,

$$\psi = \tfrac{1}{2} Ra \left(x^2 - \tfrac{1}{4}\right), \quad \theta = \phi = x, \tag{8}$$

and therefore we now introduce into Eqs. 5 small-amplitude disturbances of the following form, which are periodic in the vertical direction:

$$(\psi, \theta, \phi) = \left(\tfrac{1}{2} Ra \left(x^2 - \tfrac{1}{4}\right), x, x\right) + e^{\lambda t + iky}\Big(\Psi(x), \Theta(x), \Phi(x)\Big). \tag{9}$$

Therefore, the reduced disturbance quantities satisfy the following equations,

$$\Psi'' - k^2\Psi = Ra\,\Theta', \tag{10a}$$

$$\lambda\Theta = \Theta'' - k^2\Theta - ikRa\,x\Theta + ik\Psi + H(\Phi - \Theta), \tag{10b}$$

$$\alpha\lambda\Phi = \Phi'' - k^2\Phi + H\gamma(\Theta - \Phi), \tag{10c}$$

which are subject to the boundary conditions,

$$\Psi = \Theta = \Phi = 0 \quad \text{on} \quad x = \pm 1/2. \tag{11}$$

We may now use Eq. 10b to determine $\Psi$ in terms of $\Theta$ and $\Phi$, and to substitute it into Eq. 10a. This yields,

$$\lambda\Big(\Theta'' - k^2\Theta\Big) = \Big(\Theta'''' - 2k^2\Theta'' + k^4\Theta\Big) + ikRa\Big((x\Theta')' - k^2x\Theta\Big) + H\Big(\Theta'' - k^2\Theta\Big) \\ -H\Big(\Phi'' - k^2\Phi\Big). \tag{12}$$

We now need to find the second derivative of Eq. 10b with respect to $x$ and to subtract from that the product of Eq. 10c with $k^2$. This gives,

$$\alpha\lambda\Big(\Phi'' - k^2\Phi\Big) = \Big(\Phi'''' - 2k^2\Phi'' + k^4\Phi\Big) + H\gamma\Big(\Phi'' - k^2\Phi\Big) - H\gamma\Big(\Theta'' - k^2\Theta\Big). \tag{13}$$

Given that $\Phi$, $\Theta$ and $\Phi$ are complex quantities, we may now (i) multiply Eq. 12 by $\gamma\overline{\Theta}$ and integrate over the channel width, (ii) multiply Eq. 13 by $\overline{\Phi}$ and integrate in the same way, and (iii) sum the resulting integrals. This process yields the following,

$$\lambda \int_{-1/2}^{1/2} \left[\gamma\left(|\Theta'|^2 + k^2|\Theta|^2\right) + \alpha\left(|\Phi'|^2 + k^2|\Phi|^2\right)\right] \mathrm{d}x$$

$$= -\gamma \int_{-1/2}^{1/2} \left(|\Theta''|^2 + 2k^2|\Theta'|^2 + k^4|\Theta|^2\right) \mathrm{d}x - \alpha \int_{-1/2}^{1/2} \left(|\Phi''|^2 + 2k^2|\Phi'|^2 + k^4|\Phi|^2\right) \mathrm{d}x$$

$$-H\gamma \int_{-1/2}^{1/2} \left(|\Theta' - \Phi'|^2 + k^2|\Theta - \Phi|^2\right) \mathrm{d}x - ik\gamma Ra \int_{-1/2}^{1/2} x\left(|\Theta'|^2 + k^2|\Theta|^2\right) \mathrm{d}x. \tag{14}$$

If we take the real part of Eq. 14, then we see that the real part of $\lambda$ must always be negative, which means that all small-amplitude disturbances decay. This is precisely the same qualitative behaviour which Gill (1969) found for the system in which local thermal equilibrium prevails.

## 4 Conclusion

We have adapted the analysis of Gill (1969) to the study of the linearised stability of convection in a vertical channel with local thermal nonequilibrium effects present, and have shown that the basic state is stable to disturbances of all wavenumbers at all values of the Darcy–Rayleigh number. This does not preclude the possibility that large-amplitude disturbances might possibly persist; this would need to be studied using a nonlinear energy method such as those used by Straughan (1988) and Qin and Kaloni (1993).

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