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DISCUSSION ON THE PAPER: VORTEX INSTABILITY OF MIXED CONVECTION BOUNDARY LAYER FLOW ADJACENT TO A NONISOTHERMAL HORIZONTAL SURFACE IN A POROUS MEDIUM WITH VARIABLE PERMEABILITY

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In this Discussion we shall reflect on various aspects of the recently published paper by Elaiw and Ibrahim (2008). Their paper considers the onset of convective instabilities of a mixed convection boundary layer induced by the combined presence of a hot horizontal surface in an unbounded porous medium and an externally imposed uniform horizontal forced convective flow. These authors also adopt a power law variation in the surface temperature and allow for the effects of nonuniform porosity caused by inefficient packing of the porous medium near the horizontal surface. The primary aim of the present Discussion is to discuss the theoretical foundations of the analysis of Elaiw and Ibrahim (2008) in order to clarify them unambiguously.

KEY WORDS: *mixed convection, nonuniform permeability, linear stability theory, vortex convection*

1. INTRODUCTION

We discuss various aspects of the paper by Elaiw and Ibrahim (2008), hereinafter referred to as EI2008. In particular, we consider (i) variable porosity effects and the consequences for permeability, thermal conductivity, and thermal diffusivity; (ii) the boundary layer approximation and how it applies to the basic state computed in EI2008; (iii) the numerical solutions; and (iv) the manner in which vortex instabilities are studied, emphasizing, in particular, the mathematical validity of the method used. We have adopted the notation of EI2008 and readers are requested to refer to that paper.

2. VARIABLE POROSITY

Porosity variations occur naturally when solid particles pack imperfectly at a solid boundary (Nield and Bejan, 1992). Citing Cheng et al. (1991), Nield and Bejan quote the following formula for the porosity:

$$\epsilon = \epsilon_\infty \Big[1 + C \exp(-Ny/d_p) \Big] \qquad (1)$$

where we note that Nield and Bejan use the variable, ϕ, for porosity. In this formula the constants C and N are roughly 1.4 and 5, respectively. The value, d_p, is the typical particle diameter and this is generally regarded as a constant for typical porous media. Nield and Bejan (1992) also quote Kozeny's equation, which relates the permeability to the porosity,

$$K = \frac{d_p^2 \epsilon^3}{\beta (1-\epsilon)^2}, \qquad (2)$$

where $\beta = 150$. These two equations may be combined easily to obtain an expression for the permeability due to nonuniform porosity. On the other hand, EI2008 uses the independent expressions

$$\epsilon = \epsilon_\infty(1+de^{-y/\gamma}), \qquad K = K_\infty(1+d^* e^{-y/\gamma}), \quad (3)$$

where the constants, d and d^*, are given by $d = 1.5$ and 3, respectively. However, the substitution of Eq. (1) into Eq. (2) yields a form which is completely different from that of Eq. (3). We also note that EI2008 cite Chandrasekhara (1985) and Ibrahim and Hassanien (2000) for support, but these papers merely use the same formulas, rather than give experimental or theoretical justification for them.

Further on in paper EI2008 the value γ is defined as follows:

$$\gamma = x/\mathrm{Pe}_x^{1/2}, \quad (4)$$

when forced convection dominates, and

$$\gamma = x/\mathrm{Pe}_x^{1/3}, \quad (5)$$

when free convection dominates. The value Pe_x is proportional to the distance from the leading edge, and therefore γ, which may be regarded as being equivalent to d_p/N, varies with x. That this is generally unphysical has not been questioned by many authors, including some cited in EI2008. Indeed, the above-quoted paper by Chandrasekhara (1985) seems to be one of the first papers to introduce this way of studying variable permeability. It is stated in EI2008 that this type of variation has been taken so that the porosity and permeability variations are functions of only the similarity variable, η. Despite being unphysical, there might have been some merit in this approach had the governing equations for the basic state been reduced to self-similar form, but the boundary layer equations here remain nonsimilar. In our view, it would have been better to have taken γ to be a constant (see Rees and Pop, 2000), although it would have been much better to have followed the formulation given in Nield and Bejan (1992).

3. THERMAL CONDUCTIVITY AND DIFFUSIVITY

EI2008 quotes the following relations for the thermal conductivity and diffusivity of the porous medium:

$$\lambda_m = \epsilon\lambda_f+(1-\epsilon)\lambda_s, \qquad \alpha_m = \alpha_\infty[\epsilon+\sigma(1-\epsilon)], \quad (6)$$

where λ denotes conductivity, α denotes diffusivity, and the subscripts m, f, and s refer to the porous medium, the fluid phase, and the solid phase, respectively. Here $\alpha_\infty = \lambda_f/(\rho c_p)_f$ is the diffusivity far from the heated surface and $\sigma = \lambda_s/\lambda_f$ is a conductivity ratio.

We believe that both of the formulas in Eq. (6) are in error. Assuming first that the expression for λ_m is correct, then the appropriate expression for the thermal diffusivity of the porous medium should be

$$\alpha_m = \frac{\epsilon\lambda_f + (1-\epsilon)\lambda_s}{\epsilon(\rho c_p)_f + (1-\epsilon)(\rho c_p)_s}, \quad (7)$$

which expresses the relationship that the diffusivity is given by the ratio of the conductivity and the heat capacity per unit volume.

However, the expression for λ_m given in Eq. (6) is also incorrect in general. When the porous medium is formed of alternating strips of solid and fluid phase, and when heat transfer takes place in a direction parallel to these strips, only then is this expression correct. Moreover, should heat transfer take place in the direction perpendicular to these strips, then λ_m is given by

$$\frac{1}{\lambda_m} = \frac{\epsilon}{\lambda_f} + \frac{1-\epsilon}{\lambda_s}. \quad (8)$$

These two formulas for λ_m were discussed at length by Nield and Bejan (1992), particularly the aspect that they represent the extreme limits for the possible effective conductivity of a randomly constituted porous medium. There are also many more recent papers which cover the topic of the stagnant conductivity of a porous medium. Two examples are the papers by Stagg (2002) and Zhang et al. (2005). Further comments and references may be found in the later editions of Nield and Bejan (1998, 2006), and the highly informative article by Cheng and Hsu (1998).

4. THE BASIC FLOW

The analysis of the basic state in EI2008 is split into two separate parts which deal, respectively, with those regimes within which either free or forced convection dominates. When free convection is induced by a uniformly hot horizontal surface, the similarity variable is proportional to $y/x^{2/3}$ [see Cheng and Chang (1976) and Riley and Rees (1985)]. The resulting induced streamwise velocity outside the boundary layer is proportional to $x^{-1/3}$. Therefore when a uniform free stream is imposed upon this configuration, then the free convectively induced velocity dominates near the leading edge but becomes subdominant far from the leading edge. Thus as the distance from the leading edge increases from zero to infinity, free convective effects give way to forced convective effects smoothly — this, we believe, is the correct interpretation of the basic state. Therefore the basic state should have been computed using a parabolic solver such as the Keller box method.

However, the analysis of the basic state in EI2008 is very confusing to follow. The parameters ξ_f and ξ_n are used to model the two regimes:

$$\xi_f = \frac{\mathrm{Ra}_x}{\mathrm{Pe}_x^{3/2}}, \qquad \xi_n = \frac{\mathrm{Pe}_x}{\mathrm{Ra}_x^{2/3}}, \tag{9}$$

Forced convection dominates when $\xi_f = 0$, and free convection dominates when $\xi_n = 0$. It is also stated that $0 \leq \xi_f \leq 1$ provides *one half of the of the total mixed convection regime* and then, a little later, that $0 \leq \xi_n \leq 1$ provides *the other half of the entire mixed convection regime*. The implication is that $\xi_f = 1$ and $\xi_n = 1$ correspond to exactly the same value of x. Although Eq. (9) shows that $\xi_f = \xi_n = 1$ is consistent, the fact that γ has been defined differently within the two regimes [see Eqs. (4) and (5)] means that the governing equations themselves are not the same at this overlap point.

The presence of the power law temperature distribution also changes the interpretation of the ξ values in terms of where the free and forced convection regimes are relative to the leading edge. When the power law exponent satisfies $n < 1/2$, then the above interpretation is correct, namely, that free convection dominates near the leading edge. When it satisfies $n > 1/2$, then free convection dominates far from the leading edge. This interpretation cannot be gleaned easily from the paper of EI2008.

5. NUMERICAL SOLUTIONS OF THE BASIC STATE

The main thrust of EI2008 is the presentation of numerical solutions for the basic state and the determination of onset criteria for convection in the form of streamwise vortices. We note that very little information has been given on the numerical methods used for these purposes. The basic flow is described as having been solved using an implicit method, although the identity of that method has not been declared. The authors have not mentioned the value they used for η_∞, although it appears to be 3 in Fig. 5, a value which is too small. Figure 5 shows velocity profiles which should tend toward unity as η becomes large. If the chosen value of η_∞ had been sufficiently large, then the velocity profiles would have had a zero slope at $\eta = \eta_\infty$. Therefore these solutions cannot be trusted, despite assertions that they compare well with other authors.

6. STABILITY ANALYSIS

The authors of EI2008 introduce a system of longitudinal roll disturbances in the classic manner. However, they assume that these rolls have no appreciable streamwise variation, something which is assumed to be consistent with the boundary layer approximation. This, too, is incorrect. Most boundary layers are nonsimilar, which means that x variations in their profiles take place. The boundary layer approximation allows for streamwise diffusion to be neglected, but the x variations that could arise due to the presence of single x derivatives in the advection terms are generally not negligible.

Rees (2001) carried out a stability analysis for the onset of longitudinal vortices for a nearly vertical surface which is maintained at a uniform temperature. The reason that a nearly vertical surface was considered is that it is only within this regime that the disturbance equations may be rigorously and correctly written down within the boundary layer approximation. Within this regime the disturbance equations themselves form a parabolic system, and therefore x variations remain significant in that they allow for disturbances to change their profile and amplitude as they evolve downstream, although streamwise diffusion remains negligible. Thus the disturbance equations form a nonsimilar system.

The analysis of Rees (2001) was motivated by the publication of an earlier review chapter by Rees (1998) in which he described in detail the mathematical difficulties associated with analyzing the stability of boundary layers in porous media. Various conclusions were made by Rees (1998) that are relevant for the present discussion. The main one is that the boundary layer approximation is applied inconsistently in very many papers on vortex instability and that this fact is hidden by the use of parameters such as the local Rayleigh number, Ra_x. Indeed, Rees (1998) uses the example of an inclined, constant temperature hot surface to show in fine detail that one of the advection terms is asymptotically larger than all of the other terms in the disturbance equations. But given that a finite (typically small) value of Ra_x is obtained as the critical distance, the presence of this anomalous term passes unnoticed.

The obvious next question is whether the analysis of the authors of EI2008 is consistent. The general conclusion of Rees (1998) is that it must be mathematically inconsistent, but this may be tested easily by examining some of the data presented as it is important to underscore the fact that approximations, when made, must be verified *a posteriori*. If we take the lowest curve in Fig. 7 of EI2008, then when $n = 0.5$, we have a critical value $\mathrm{Pe}_x \simeq 30$ when $\xi_f = 0.6$. Given that the similarity variable, η, is defined as $\eta = \mathrm{Pe}_x^{1/2} y/x$ in Eq. (9) of EI2008, and if we define the edge of the boundary layer as corresponding to $\eta = 3$, then we obtain the relation $y \simeq 0.548x$. (We note that $\xi_f = 0.6 \Rightarrow \mathrm{Ra}_x \simeq 98.6$, from which x may be found.) In other words, given that x must be the critical distance, the boundary layer thickness at that point is then just over half of the critical distance. Therefore the critical distance is well within a range of values where the equations are fully elliptic. It is clear, therefore, that the boundary layer approximation is inapplicable at such distances.

7. CONCLUSION

The manner in which the stability analysis was been carried out in EI2008 is one which continues to be used widely and it is possible to show that these are mathematically inconsistent. One example of such a demonstration is described in the above–quoted chapter by Rees (1998). However EI2008 has also adopted what we regard as an unphysical variation of the permeability near the solid surface, a feature which has also been used widely. In addition, there are some inaccuracies associated with the expressions for the thermal conductivity and diffusivity, the basic flow, and the numerical computations themselves.

The general message given by the reviews of Rees (1998, 2002a), and many of the papers cited therein, is that instability generally arises much too close to the leading edge for the boundary layer approximation to be valid. Therefore the approximate methods which were introduced into porous medium flows by Hsu et al. (1978), and which were novel and groundbreaking at the time, are now ones which should be discarded in favour of more accurate methods. There are situations where the disturbance equations may be written down consistently within the boundary layer approximation [see the studies of Rees (2001, 2002b, 2003 and 2009) and Rees et al. (2008)], but these are special cases which involve either asymptotically small or asymptotically large parameters. For other cases it is necessary to solve the fully elliptic governing equations, as in Rees and Bassom (1993) and Rees (1993), and this brings with it an entirely new set of challenges for the porous medium community. Work on this is currently being undertaken.

REFERENCES

Chandrasekhara, B. C., Convection in the presence of horizontal impermeable surfaces in saturated porous media with variable permeability, *Wärme–Stoffübertragung*, vol. **19**, pp. 195–201, 1985

Cheng, P. and Chang, I.–D., Buoyancy induced flows in a porous medium adjacent to impermeable horizontal surfaces, *Int. J. Heat Mass Transfer*, vol. **19**, pp. 1267–1272, 1976.

Cheng, P. and Hsu, C. T., Heat conduction, *Transp. Phenomena in Porous Media*, (eds. Ingham, D.B. & Pop, I.) Pergamon, UK, pp. 57–76, 1998.

Cheng, P., Chowdhury, A., and Hsu, C. T., Forced convection in packed tubes and beds with variable porosity and thermal disperson effects, *Convective Heat and Mass Transfer in Porous Media*, (eds. Kakaç, S., Kilkiş, B., Kulacki, F. A., and Arinç, F.) Kluwer Academic, Dordrecht, pp. 625–653, 1991.

Elaiw, A. M. and Ibrahim, F. S., Vortex instability of mixed convection boundary layer flow adjacent to a nonisothermal horizontal surface in a porous medium with variable permeability, *J. Porous Media*, vol. **11**, pp. 305–321, 2008, referred to as EI2008.

Hsu, C. T., Cheng, P., and Homsy, G. M., Instability of free convection flow over a horizontal impermeable surface in a porous medium, *Int. J. Heat Mass Transfer*, vol. **21**, pp. 1221–1228, 1978.

Ibrahim, F. S. and Hassanian, I. A., Influence of variable permeability on combined convection along a non-isothermal wedge in saturated porous medium, *Transp. Porous Media*, vol. **39**, pp. 57–71, 2000.

Nield, D. A. and Bejan, A., *Convection in Porous Media*, New York, Springer, 1992 (1st ed.), 1998 (2nd ed.), 2006 (3rd ed.).

Rees, D. A. S., A numerical investigation of the nonlinear wave stability of vertical thermal boundary layer flow in a porous medium, *J. Appl. Math. Phys. (Z.A.M.P.)*, vol. **44**, pp. 306–313, 1993.

Rees, D. A. S., The effect of inertia on free convection from a horizontal surface embedded in a porous medium, *Int. J. Heat Mass Transfer*, vol. **39**, pp. 3425–3430, 1996.

Rees, D. A. S., Thermal boundary layer instabilities in porous media: a critical review, *Transp. Phenomena in Porous Media*, (eds. D.B. Ingham & I. Pop) Pergamon, pp. 233–259, 1998.

Rees, D. A. S., Vortex instability from a near-vertical heated surface in a porous medium, I Linear theory, *Proc. Roy. Soc. Lond.*, vol. **A457**, pp. 1721–1734, 2001.

Rees, D. A. S., Recent advances in porous medium thermal boundary layer instabilities, *Transp. Phenomena in Porous Media II*, (eds. D.B. Ingham and I. Pop.) Pergamon, pp. 54–81, 2002a.

Rees, D. A. S., Vortex instability from a near-vertical heated surface in a porous medium, II Nonlinear evolution, *Proc. Roy. Soc. Lond.*, vol. **A458**, pp. 1575–1592, 2002b.

Rees, D. A. S., Nonlinear vortex development in free convective boundary layers in porous media, *NATO ASI Proc., Neptun, Romania, June 9–20 2003*, pp. 449–458, 2003.

Rees, D. A. S., The onset and nonlinear development of vortex instabilities in a horizontal forced convection boundary layer with uniform surface suction, *Transp. in Porous Media* R.A.Wooding Special Issue, vol. **77**, No. 2, pp. 243–265, 2009.

Rees, D. A. S. and Bassom, A. P., The nonlinear nonparallel wave instability of free convection induced by a horizontal heated surface in fluid-saturated porous media, *J. Fluid Mech.*, vol. **253**, pp. 267–296, 1993.

Rees, D. A. S. and Pop, I., Vertical free convection in a porous medium with variable permeability effects, *Int. J. Heat Mass Transfer*, vol. **43**, pp. 2565–2571, 2000.

Rees, D. A. S, Postelnicu, A., and Bassom, A. P., The linear vortex instability of the near-vertical line source plume in porous media, *Transp. Porous Media*, vol. **74**, pp. 221–238, 2008.

Riley, D. S. and Rees, D. A. S., Non–Darcy natural convection from arbitrarily inclined heated surfaces in saturated porous media, *Quart. J. Mech. Appl. Math.*, vol. **38**, pp. 277–295, 1985.

Stagg, J. E. J., Estimating the thermal conductivity of chars and porous residues using thermal resistor networks, *Fire Safety J.*, vol. **37**, pp. 107–119, 2002.

Zhang, H., Ge, X., and Ye, H., Effectiveness of the heat conduction reinforcement of particle filled composites, *Model. Simul. Mater. Sci.*, vol. **13**, pp. 401–412, 2005.