Transp Porous Med (2008) 74:221–238
DOI 10.1007/s11242-007-9195-3

The Linear Vortex Instability of the Near-vertical Line Source Plume in Porous Media

D. Andrew S. Rees · Adrian Postelnicu · Andrew P. Bassom

Received: 31 August 2007 / Accepted: 29 November 2007 / Published online: 8 January 2008

Abstract Free convection plumes usually rise vertically, but do not do so when in an asymmetrical environment. In such cases they are susceptible to a thermoconvective instability because warmer fluid lies below cooler fluid in the upper half of the plume. We analyse the behaviour of streamwise vortex disturbances in plumes that are close to being vertical. The linearised equations subject to the boundary layer approximation are parabolic and are solved using a marching method. Our computations indicate that disturbances tend to be centred in the upper half of the plume. A neutral curve is determined and an asymptotic theory is developed to describe the right hand branch of this curve. The left hand branch is not amenable to an asymptotic analysis, and it is found that the onset of convection for small wavenumbers is very sensitively dependent on both the profile of the initiating disturbance and where it is introduced.

Keywords Free convection · Line source · Plume · Vortex instability

Nomenclature
c Constant in Eq. 2.16
C_p Specific heat
d Natural lengthscale
f Reduced streamfunction

D. A. S. Rees
Department of Mathematics, University of Bristol, Bristol BS8 1TW, UK

D. A. S. Rees (✉)
Department of Mechanical Engineering, University of Bath, Bath BA2 7AY, UK
e-mail: ensdasr@bath.ac.uk

A. Postelnicu
Department of Thermal Engineering and Fluid Mechanics, Transilvania University, Bd. Eroilor 29, Brasov 500036, Romania

A. P. Bassom
School of Mathematics and Statistics, University of Western Australia, Crawley, WA 6009, Australia

g	Acceleration due to gravity
k	Vortex wavenumber
K	Permeability
p	Fluid pressure
P	Perturbation pressure
q'''	Strength of heat source
T	Temperature
u, v, w	Darcy velocities in the x, y and z-directions
U, V, W	Perturbation velocities
x, y, z	Cartesian coordinates
Y	Scaled form of η in asymptotic analysis

Greek symbols

α	Thermal diffusivity
β	Coefficient of cubical expansion
δ	Orientation of the plume centreline
ϵ	Inclination of a horizontal boundary
μ	Fluid viscosity
θ	Non-dimensional temperature
Θ	Perturbation temperature
ϕ	Angular coordinate
ϕ^+, ϕ^-	Orientations of bounding surfaces
ψ	Streamfunction
ρ	Fluid density
η	Similarity variable
ξ	Scaled value of x

Other symbols, subscripts and superscripts

$-$	Dimensional
$\hat{}$	Non-dimensional
$'$	Derivative with respect to η
c	Critical value
∞	Ambient
$0, 1, 2, \ldots$	Terms in asymptotic series

1 Introduction

Free convection plumes within porous media arise in a number of important contexts, notably above buried electrical cables or hot pipes. Further, the presence in landfill sites of toxic substances, such as radioactive waste or chemical pollutants, may also cause a solutal plume to form whether or not there is a background groundwater flow. The dispersal of pollutants in particular has important ramifications environmentally, and should the resulting plume be unstable, then this becomes an extra means by which solutes may spread. In the present article, then, we address the issue of the instability of a free convection plume.

In a fully idealised setting of an infinitely large porous region, a plume with a vertical centreline is formed above a uniform horizontal line source of heat, and it assumes a boundary-layer form at sufficiently large distances above the source. Thus, to a high degree of accuracy, the resulting temperature field and the induced flow may be described by a classical boundary layer theory. Wooding (1963) was the first to consider this type of flow and he showed that the centreline temperature of the line source plume decays as $x^{-1/3}$, where x is the distance from the source along the centreline, while the thickness of the plume is proportional to $x^{2/3}$. In fact, the boundary layer equations for the plume admit analytical solutions.

Wooding's work has been extended in a variety of ways. Ingham (1988) considered the situation where form drag is dominant. Although the resulting boundary layer equations are then more complicated, they still admit an analytical solution. Rees and Hossain (2001) considered the transition between the region within which form drag dominates (which is relatively close to the line source but still sufficiently distant that the boundary layer approximation applies) and the Darcy regime at larger distances. On the other hand, Afzal (1985) sought to determine the Darcy-flow plume solution more accurately by employing a higher order boundary layer theory, where the flow in the region external to the plume influences higher order terms in the solution for the plume. This analysis furnishes small corrections to Wooding's (1963) results.

Afzal's analysis is restricted to a symmetrical domain where the line source is placed at the apex of a wedge of saturated porous medium which has a vertical centreline. This free convection plume still continues to rise vertically, just as one would expect intuitively. Bassom et al. (2001) relaxed this restriction by considering a porous medium bounded by two plane surfaces which are located arbitrarily, but are still such that the line source is located at the horizontal intersection of the planes; this is, of course, equivalent to a rotation of Afzal's wedge about the line source. Bassom et al. (2001) used a higher-order boundary layer theory to derive an analytical formula which relates the inclinations of the bounding surfaces to the angle that the centreline of the plume makes to the vertical. Rees et al. (2002) showed that anisotropy has the same effect, even in an unbounded porous domain.

When a plume rises at an angle to the vertical, the upper half is unstably stratified and may therefore be susceptible to thermoconvective instability, as relatively warm fluid lies below relatively cool fluid in the upper half of the plume. Given Wooding's observations alluded to above, a Darcy-Rayleigh number which is based upon the width of the plume as the lengthscale, and the temperature variation across the plume, will increase as $x^{1/3}$ as x increases. This suggests that the plume should become increasingly unstable with distance from the source.

Noting that there is no reference to instability in the seven pages on free convection plumes in the monograph by Nield and Bejan (2006), this article is devoted to making the first step in analysing such an instability. We follow closely the analysis of Rees (2001) which was concerned with the vortex instability of the boundary layer induced by a near-vertical constant temperature surface. Although the effect of an asymmetrical domain is to cause the plume to rise at an $O(1)$ angle to the vertical, the chief advantage of considering the near-vertical plume is that the linearised stability equations satisfy the boundary layer approximation with no further simplifying assumptions being necessary. These equations form a system of parabolic partial differential equations. By contrast, for plumes with an $O(1)$ inclination, it would be necessary to solve a fully elliptic system. In the present context, a plume which rises almost vertically corresponds to one where, for example, the line source is embedded in a flat surface which is aligned at a small angle to the horizontal.

We shall seed vortex disturbances at locations which are relatively close to the line source, and use the solutions for different wavenumbers to construct a neutral stability curve. Other detailed aspects of the solutions obtained are also discussed.

2 Equations of Motion and Basic Flow

We consider the stability of a line source plume in a porous medium within which Darcy's law and the Boussinesq approximation are assumed to be valid, and where the fluid and solid phases are in Local Thermal Equilibrium. The steady equations governing mass conservation, fluid motion and heat transport are

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} + \frac{\partial \overline{w}}{\partial \overline{z}} = 0, \tag{2.1a}$$

$$\overline{u} = \frac{K}{\mu}\left[-\frac{\partial \overline{p}}{\partial \overline{x}} + \rho_\infty g\beta(\overline{T} - \overline{T}_\infty)\cos\delta\right], \tag{2.1b}$$

$$\overline{v} = \frac{K}{\mu}\left[-\frac{\partial \overline{p}}{\partial \overline{y}} + \rho_\infty g\beta(\overline{T} - \overline{T}_\infty)\sin\delta\right], \tag{2.1c}$$

$$\overline{w} = -\frac{K}{\mu}\frac{\partial \overline{p}}{\partial \overline{z}}, \tag{2.1d}$$

$$\overline{u}\frac{\partial \overline{T}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{T}}{\partial \overline{y}} + \overline{w}\frac{\partial \overline{T}}{\partial \overline{z}} = \alpha\nabla^2\overline{T}. \tag{2.1e}$$

Here $\overline{x}$, $\overline{y}$ and $\overline{z}$ are the streamwise, cross–stream and spanwise Cartesian coordinates, $\overline{u}$, $\overline{v}$ and $\overline{w}$ denote the corresponding fluid flux velocities, the pressure is $\overline{p}$ and the temperature $\overline{T}$. In system (2.1) the angle of inclination of the x-axis from the vertical is δ which is also taken to be the inclination of the centreline of the plume; the overall configuration is illustrated in Fig. 1 where ϕ is the anticlockwise polar angle from the upward vertical. The porous medium is therefore wedge-shaped and of infinite extent, and the boundaries of the wedge are assumed to be impermeable and are held at the ambient temperature T_∞.

The equation for the global conservation of heat takes the form,

$$q''' = \rho_\infty C_p \int_{-\infty}^{\infty} \overline{u}(\overline{T} - \overline{T}_\infty)\, d\overline{y} \tag{2.2}$$

where q''' is the rate of heat production per unit length of the line source. We have already neglected the conduction component in (2.2) as that is easily shown to be asymptotically small relative to the advective term when the boundary layer approximation is valid.

Equation (2.1) may be non-dimensionalised using the natural lengthscale,

$$d = \frac{(\rho_\infty C_p)\mu\alpha^2}{\rho_\infty g\beta K q'''}. \tag{2.3}$$

Therefore we introduce the substitutions,

$$(\overline{x}, \overline{y}, \overline{z}) = d(\hat{x}, \hat{y}, \hat{z}), \quad (\overline{u}, \overline{v}, \overline{w}) = \frac{\alpha}{d}(\hat{u}, \hat{v}, \hat{w}), \tag{2.4a}$$

$$\overline{p} = \frac{\mu\alpha}{K}\hat{p}, \quad \overline{T} = \overline{T}_\infty + \frac{q'''}{\rho_\infty C_p\alpha}\hat{T}. \tag{2.4b}$$

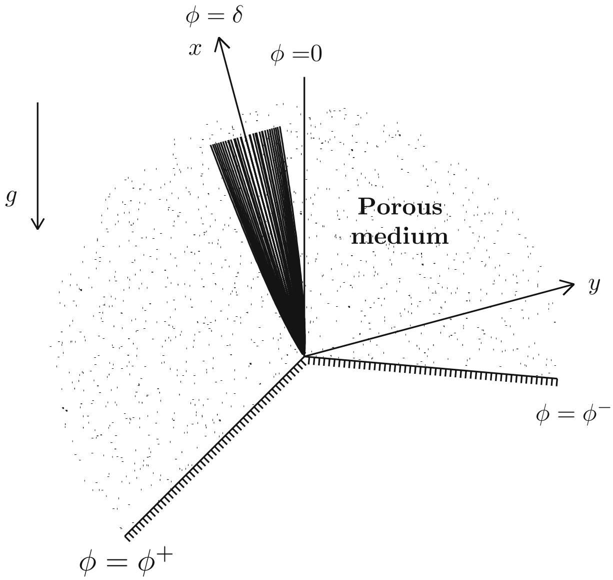


Fig. 1 Showing the coordinate system, the locations of the bounding surfaces ($\phi = \phi^-, \phi^+$) and the direction of gravity. In this sketch ϕ is the polar angle measured anticlockwise from the upward vertical. The line source is located at the origin

Equations (2.1) and (2.2) become,

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} = 0, \tag{2.5a}$$

$$\hat{u} = -\frac{\partial \hat{p}}{\partial \hat{x}} + \hat{T}\cos\delta, \quad \hat{v} = -\frac{\partial \hat{p}}{\partial \hat{y}} + \hat{T}\sin\delta, \quad \hat{w} = -\frac{\partial \hat{p}}{\partial \hat{z}}, \tag{2.5b, c, d}$$

$$\hat{u}\frac{\partial \hat{T}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{T}}{\partial \hat{y}} + \hat{w}\frac{\partial \hat{T}}{\partial \hat{z}} = \nabla^2 \hat{T}. \tag{2.5e}$$

and

$$\int_{-\infty}^{\infty} \hat{u}\hat{T}\, d\hat{y} = 1. \tag{2.6}$$

Following Rees (2001) it is necessary to introduce one further scaling based upon the fact that the centreline of the plume is almost vertical ($\delta \ll 1$). We set

$$\hat{x} = \left(\frac{\cos^2\delta}{\sin^3\delta}\right) x, \quad (\hat{y}, \hat{z}) = \left(\frac{\cos\delta}{\sin^2\delta}\right)(y, z), \tag{2.7a}$$

$$\hat{u} = (\sin\delta) u, \quad (\hat{v}, \hat{w}) = \left(\frac{\sin^2\delta}{\cos\delta}\right)(v, w), \tag{2.7b}$$

$$\hat{p} = p, \quad \hat{T} = \left(\frac{\sin\delta}{\cos\delta}\right)\theta, \tag{2.7c}$$

whereupon system (2.5) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.8a}$$

$$u = \theta - \left(\frac{\sin^2\delta}{\cos^2\delta}\right)\frac{\partial p}{\partial x}, \quad v = \theta - \frac{\partial p}{\partial y}, \quad w = -\frac{\partial p}{\partial z}, \tag{2.8b}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = \left(\frac{\sin^2\delta}{\cos^2\delta}\right)\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \tag{2.8c}$$

and the heat flux condition reduces to

$$\int_{-\infty}^{\infty} u\theta \, dy = 1. \tag{2.9}$$

Thus we see that small values of δ are equivalent to invoking the boundary layer approximation, since the only terms in (2.8) which are then formally negligible are the streamwise diffusion terms.

We may take the line source to be embedded in a plane-bounding surface at a small inclination, ϵ, away from the horizontal. In Fig. 1 this is equivalent to setting

$$\phi^+ = \tfrac{1}{2}\pi + \epsilon, \quad \phi^- = -\tfrac{1}{2}\pi + \epsilon. \tag{2.10}$$

The analysis of Bassom et al. (2001) yields the formula,

$$\cot\left(\frac{\phi^+ - \delta}{3}\right) - \cot\left(\frac{\delta - \phi^-}{3}\right) = -2\tan\delta \tag{2.11}$$

which relates the inclination of the plume centreline, δ, to the inclinations of the bounding surfaces. When ϵ is small in Eq. 2.10, it is straightforward to show that

$$\delta \sim \tfrac{4}{7}\epsilon, \tag{2.12}$$

and therefore a slight inclination of the bounding surface away from the horizontal causes a small change in the direction taken by the plume. It is this change that renders the plume subject to instability.

On defining the streamfunction, ψ, in the usual way: $u = \psi_y$, $v = -\psi_x$, Eqs. 2.8 and 2.9 admit the well-known solutions (see Wooding 1963),

$$\psi = x^{1/3} f(\eta), \quad \theta = x^{-1/3} f'(\eta), \tag{2.13}$$

where $\eta = y/x^{2/3}$ is the similarity variable, and where f satisfies,

$$f''' + \tfrac{1}{3}(ff'' + f'f') = 0, \tag{2.14}$$

subject to

$$f(0) = 0, \quad f''(0) = 0, \quad \int_{0}^{\infty} f'f' \, d\eta = \tfrac{1}{2}. \tag{2.15}$$

Ingham's (1988) analytical solution is

$$f = c\tanh(\tfrac{1}{6}c\eta), \quad c = (9/2)^{1/3}. \tag{2.16}$$

We may now determine expressions for the velocities, the temperature and the pressure:

$$u = \theta = x^{-1/3}f', \quad v = x^{-2/3}\left[\tfrac{2}{3}\eta f' - \tfrac{1}{3}f\right], \quad \frac{\partial p}{\partial \eta} = x^{1/3}f' + \left[\tfrac{1}{3}f - \tfrac{2}{3}\eta f'\right]. \tag{2.17}$$

3 Stability Analysis

From this point onwards we shall use lowercase notation for the dependent variables corresponding to the basic flow described above, and upper case for the perturbations. Therefore we introduce disturbances by replacing p in Eqs. 2.8 by $p+\varepsilon P$ where $|\varepsilon| \ll 1$ and where the latter p is the basic pressure solution. Similar substitutions are made for the other dependent variables. The linearised perturbation equations are found to be:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \tag{3.1}$$

$$U = \Theta, \quad V = -\frac{\partial P}{\partial y} + \Theta, \quad W = -\frac{\partial P}{\partial z}, \tag{3.2a–c}$$

$$\frac{\partial^2 \Theta}{\partial y^2} + \frac{\partial^2 \Theta}{\partial z^2} = u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} + \frac{\partial \theta}{\partial x}U + \frac{\partial \theta}{\partial y}V, \tag{3.2d}$$

where we have now invoked the boundary layer approximation by taking the limit $\delta \to 0$. After substitution of the basic flow variables (u, v, p and θ), as given above, the introduction of the coordinate transformation,

$$\eta = y/x^{2/3}, \quad \xi = x^{1/3}, \tag{3.3}$$

and the slight rescaling,

$$P \to x^{1/3}P, \tag{3.4}$$

which is made for convenience, the perturbation equations become

$$P'' + \xi^4 P_{zz} = \left[\tfrac{1}{3}\xi\Theta_\xi - \tfrac{2}{3}\eta\Theta'\right] + \xi\Theta', \tag{3.5a}$$

$$\Theta'' + \xi^4\Theta_{zz} = \left(\tfrac{1}{3}f'\right)[\xi\Theta_\xi - \Theta] - \left(\tfrac{1}{3}f\right)\Theta' + \left(\xi - \tfrac{2}{3}\eta\right)f''\Theta - f''P'. \tag{3.5b}$$

Finally we may now introduce vortex rolls using,

$$(P, \Theta) \to (P, \Theta)\cos kz, \tag{3.6}$$

and derive the system

$$P'' - \xi^4k^2P = [\tfrac{1}{3}\xi\Theta_\xi - \tfrac{2}{3}\eta\Theta'] + \xi\Theta', \tag{3.7a}$$

$$\Theta'' - \xi^4k^2\Theta = \left(\tfrac{1}{3}f'\right)\left[\xi\Theta_\xi - \Theta\right] - \left(\tfrac{1}{3}f\right)\Theta' + \left(\xi - \tfrac{2}{3}\eta\right)f''\Theta - f''P'. \tag{3.7b}$$

The boundary conditions are that

$$P, \ \Theta \ \to 0 \quad \text{as} \quad \eta \to \pm\infty. \tag{3.8}$$

This is a set of parabolic equations in which ξ denotes the direction of marching. We follow Rees (2001) by introducing thermal vortex disturbances of various shapes at different values of ξ (which we shall call ξ_i) and follow their evolution with ξ. Our main set of numerical results arise from considering the thermal disturbance $\Theta = \exp(-\eta^2)$ at $\xi_i = 1$. The system was solved using a fairly standard implementation of the Keller box method (see Rees 2001). We used 500 equally spaced intervals in the range $-25 \leq \eta \leq 25$, and a steplength of 0.005 in the ξ-direction. Thus quite highly accurate results were obtained.

Rees (2001) studied convection which was induced by a uniformly hot surface. In the presence of instability the overall rate of heat transfer from the surface is increased and the thermal energy together with the surface rate of heat transfer were used as separate measures of the onset of instability. Here we choose a different way of monitoring the growth of vortices as with the plume extra heat cannot be generated away from the line source and thermal energy must be conserved. Therefore we choose to use the maximum temperature of the disturbance to monitor whether or not the plume is unstable.

4 Results and Discussion

4.1 General Characteristics and the Neutral Curve

We begin by showing how vortex disturbances evolve after their inception. Figure 2 shows the detailed isotherms for two cases where the point of introduction of the disturbance is $\xi_i = 1$ and where the wavenumbers are $k = 0.007$ and $k = 0.004$. It is important to recall that these isotherms represent a cross-section through the full three-dimensional disturbance, as given by Eq. 3.6.

In both cases the disturbance decays at first, and moves upwards so that it is placed well within the upper half of the plume, which is where the plume is unstably stratified. The thermal disturbance continues to penetrate into the lower, stably stratified, half of the plume, but it is much weaker there. When $k = 0.007$ the maximum disturbance temperature ceases to decay at $\xi \simeq 6.14$, and it enjoys a short interval of growth until $\xi \simeq 8.70$, after which point it decays again. Such a pattern arises in general if the wavenumber is not too large. For $k = 0.004$ the interval of growth is $6.66 < \xi < 14.34$.

Figure 3 shows how the maximum temperature of the disturbance varies with ξ for a chosen set of wavenumbers. It is clear that the interval within which the disturbance grows depends strongly on the wavenumber of the vortices. The curve for $k = 0.008$ displays only a very short interval of growth, whereas the $k = 0.009$ and $k = 0.010$ curves decay for all values of ξ.

On finding the maxima and minima of the curves shown in Fig. 3 and the corresponding curves for many other intermediate values of k, it is possible to construct a neutral stability curve to summarise the linear stability properties of the present system. This is shown in Fig. 4 where the critical values of ξ are denoted by ξ_c. Careful computations using ξ-steps as small as 0.001 suggest that the maximum wavenumber for which instability may arise is $k \simeq 0.00833$, and that the disturbances with a larger wavenumber always decay. On the other hand, the smallest value for ξ_c corresponding to neutral stability arises when

$$k_c \simeq 0.00691 \quad \text{and} \quad \xi_c \simeq 6.014. \tag{4.1}$$

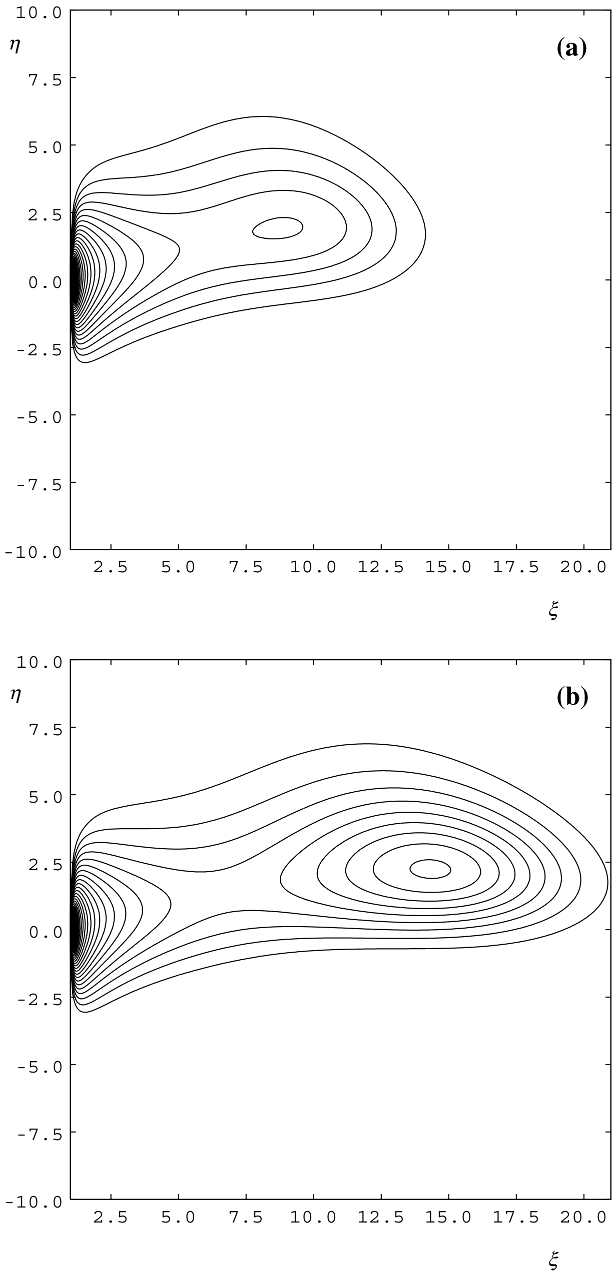


Fig. 2 Displaying disturbance isotherms using 40 equally spaced intervals. The initial disturbance, $\Theta = e^{-\eta^2}$ was introduced at $\xi_i = 1$. (**a**) $k = 0.007$, (**b**) $k = 0.004$

We have also determined an approximate neutral curve by solving the ordinary differential eigenvalue problem for ξ and k which is obtained by setting all ξ-derivatives to zero in Eqs. 3.7. Calculations were carried out using a fourth order Runge-Kutta scheme with the shooting method, and at least four significant figures of accuracy were obtained. We see that

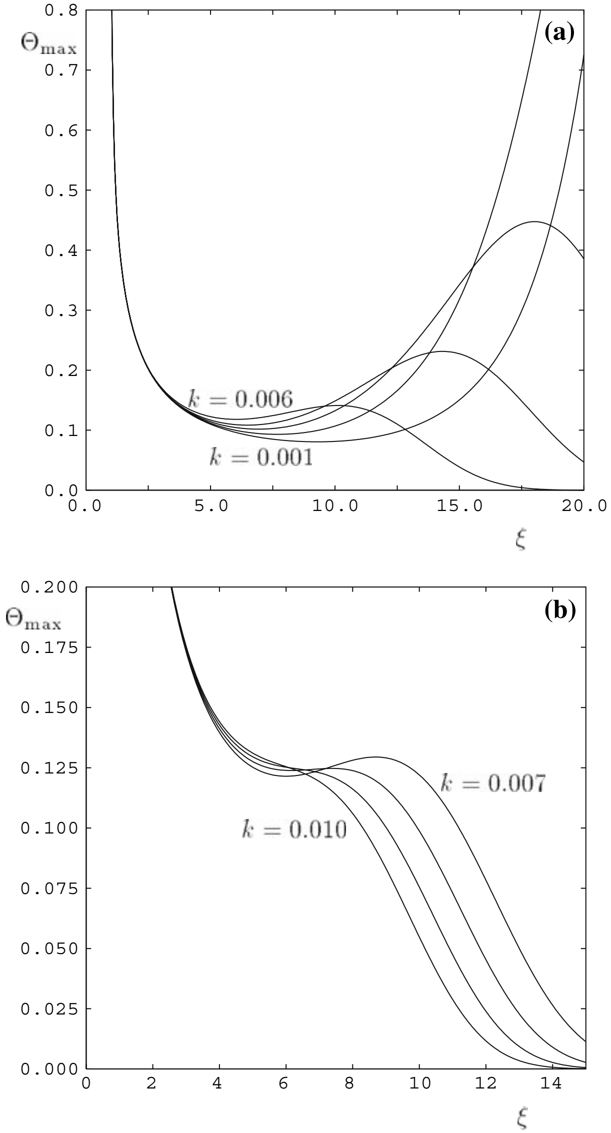


Fig. 3 The variation of Θ_{max} with ξ for (**a**) $k = 0.001$, 0.002, 0.003, 0.004, 0.005 and 0.006; (**b**) $k = 0.007$, 0.008, 0.009 and 0.0100. The initial disturbance, $\Theta = e^{-\eta^2}$ was introduced at $\xi_i = 1$

this curve lies within the neutral curve obtained from the parabolic simulations; the reason for this is that the zero ξ-derivative assumption over-constrains the equations, resulting in the need for a larger local Rayleigh number for instability.

For the sake of comparison with the parabolic simulations, we have found the minimum point on the approximate neutral curve that is

$$k_c \simeq 0.0059014, \quad \text{and} \quad \xi_c \simeq 6.61454. \tag{4.2}$$

Thus we see that the critical wavenumber is poorly predicted by the approximate analysis.

The disturbance temperature profile corresponding to the minimum point given by (4.1) is displayed in Fig. 5. This profile confirms that the disturbance is centred above $\eta = 0$,

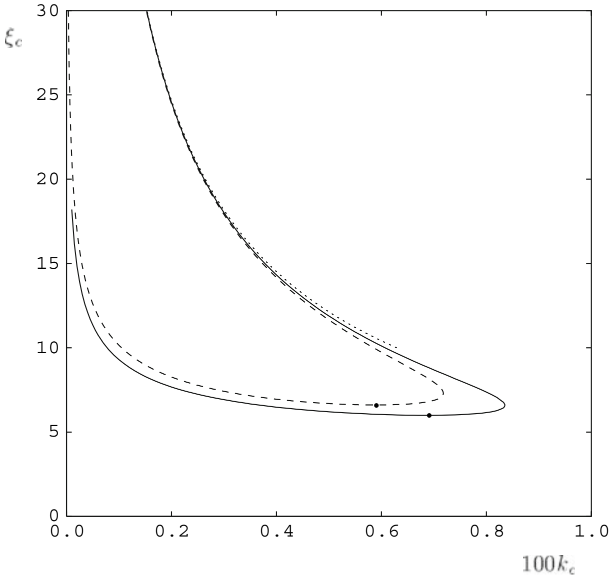


Fig. 4 Neutral curve for the onset of convection based on $\xi_i = 1$. Vortex disturbances grow in the region between the upper and lower branches of the curve. Continuous line: parabolic simulations; dashed line: approximate solution; dotted line: three-term asymptotic solution for the right hand branch. The bullets denote the respective minima

the centreline of the plume. In addition we see that there is a small region with the opposite sign when η is negative; this region of opposite sign is also present in the data used to create Fig. 2, but it is too weak to be seen there.

4.2 The Effect of Varying the Value of ξ_i and the Initiating Disturbance Profile

In all the results displayed so far, the initiating disturbance temperature profile is $\exp(-\eta^2)$ and it has been introduced at $\xi_i = 1$, which is well upstream of the minimum value of ξ_c given in (4.1). Rees (2001) discusses at length the impact on neutral stability of using different initial disturbance profiles and different positions at which the disturbance is initiated. The general conclusion that was drawn in that article is that the neutral curve is independent of these factors whenever the disturbance is introduced at sufficiently small values of ξ_i. It is therefore important to determine whether such a conclusion also applies for the present plume.

Figure 6 shows how the evolution of the maximum disturbance temperature, $\Theta_{\max}$, changes when the position at which the disturbance is introduced is altered. The wavenumber is taken to be $k = 0.004$, for which the neutral location shown in Fig. 4 is $\xi = 6.47$. The actual computed data is displayed in Fig. 6a, and black circles are used to denote where $\Theta_{\max}$ attains maximum or minimum values. In this figure the disturbances are introduced at $\xi = 0.1$, $\xi = 0.5$ and integer values of ξ from 1 to 6. This figure shows that there is no significant change in the critical value of ξ when the disturbance is introduced upstream of $\xi = 3$. When the disturbance is introduced further downstream then the critical value also moves downstream.

Fig. 5 Disturbance temperature profile at the critical point given by Eq. 4.2

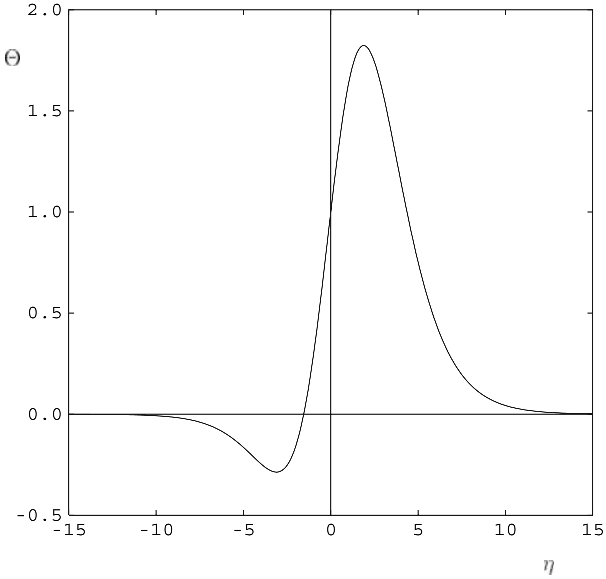


Figure 6b shows the same curves, but here Θ_{max} has been normalised to give the same value at $\xi = 20$; such a procedure is valid since we are considering linearised theory, and the absolute magnitude of the disturbance is not important. The normalisation allows us to see that the variation of Θ_{max} follows a common path once transients have decayed.

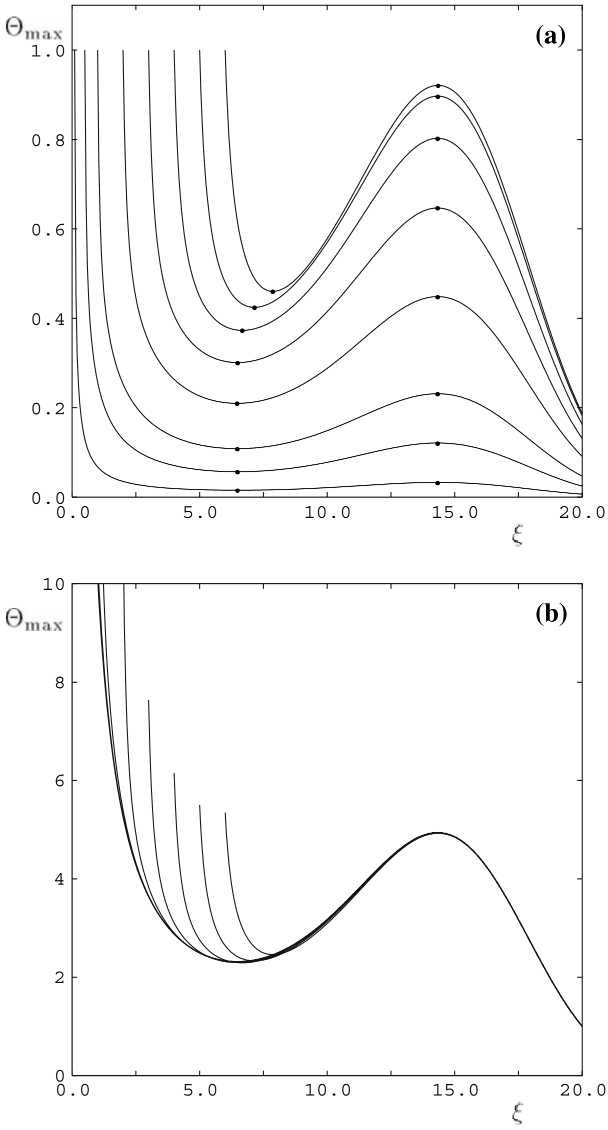
So far, then, the present results indicate that the near-vertical plume has the same qualitative stability characteristics as the near-vertical heated surface of Rees (2001), namely that the neutral curve is independent of ξ_i if ξ_i is sufficiently small.

However, when we employ a vortex disturbance with a smaller wavenumber, the situation changes. Figure 7 shows the equivalent of Fig. 6a for $k = 0.0001$. In this figure we see that ξ_c initially remains roughly constant as ξ_i increases from very small values. However, as ξ_i increases further, the critical distance, ξ_c, decreases before eventually increasing again. This latter variation with ξ_i is understandable from the point of view that the plume is clearly unstable, but it takes only a small adjustment of the disturbance profile to obtain one that grows. However, the reduction in ξ_c is more difficult to explain. We think that the natural evolution of a disturbance after its introduction at ξ_i is such that it does not naturally yield the most quickly growing disturbance at any chosen value of ξ when ξ_i is small. However, at larger values of ξ_i, the chosen initial disturbance shape is more amenable to growth, as there must be a wide range of growing profiles that are available, and this gives rise to onset closer to the line source.

The ξ_c data shown in Fig. 7 may also be plotted against ξ_i to show more clearly the dependence of the former on the latter. Figure 8 shows such curves for a variety of wavenumbers and for three different initial disturbance profiles: $\Theta = \exp[-(\eta - \eta_i)^2]$, where η_i takes the three values, 0, 1 and 2.393072, a value which arises in the asymptotic analysis presented in the next section.

In this figure we see the same general behaviour that was described in relation to Fig. 7, but the severity of the reduction in ξ_c as ξ_i varies depends very strongly on the wavenumber k. Indeed, when k is close to the critical value given by Eq. 4.1, ξ_c is roughly constant (with only a very slight reduction) before it begins to rise roughly linearly. On the other hand, for

Fig. 6 Showing the effect on the evolution of $\Theta_{\max}$ of varying the position at which the disturbance, $\Theta = e^{-\eta^2}$ is introduced: (**a**) the computed values of $\Theta_{\max}$; (**b**) values that are scaled be to equal at $\xi = 20$. Black circles denote maxima and minima. The wavenumber is $k = 0.004$



small wavenumbers, there is a large amount of reduction in ξ_c from its small-ξ_i value, before it too begins to rise. However, the degree of reduction also depends very strongly on η_i, which is where the initial disturbance is centred. When k is relatively large and η_i is positive, then $\xi_i \ll 1$ yields the smallest ξ_c. On the other hand there is substantial variation in ξ_c with η_i when k is small.

We have not presented equivalent curves for negative values of η_i. The reason is that the evolution of such a disturbance does not follow the general pattern indicated in Fig. 2. Rather, the disturbance generates a second vortex above itself while the lower vortex eventually decays. Thus a stability criterion which is based on a maximum temperature, or even a maximum absolute temperature, is fraught with difficulty of interpretation.

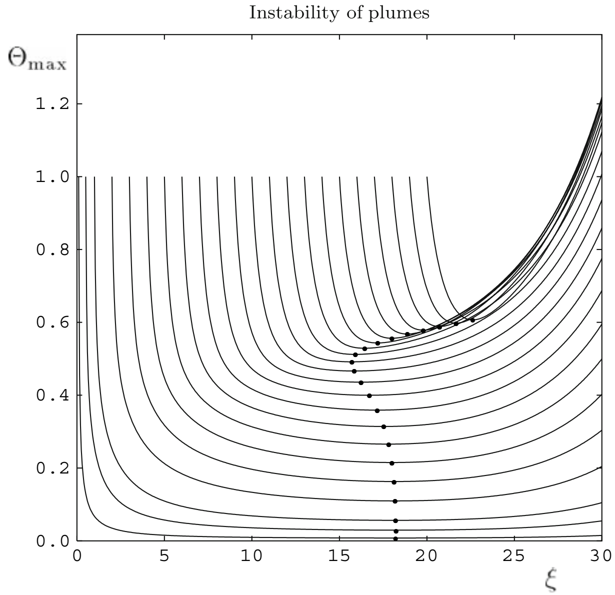


Fig. 7 Showing the effect on the evolution of Θ_{max} of varying the position, ξ_i, at which the disturbance, $\Theta = e^{-\eta^2}$, is introduced. Black circles denote ξ_c. The wavenumber is $k = 0.0001$

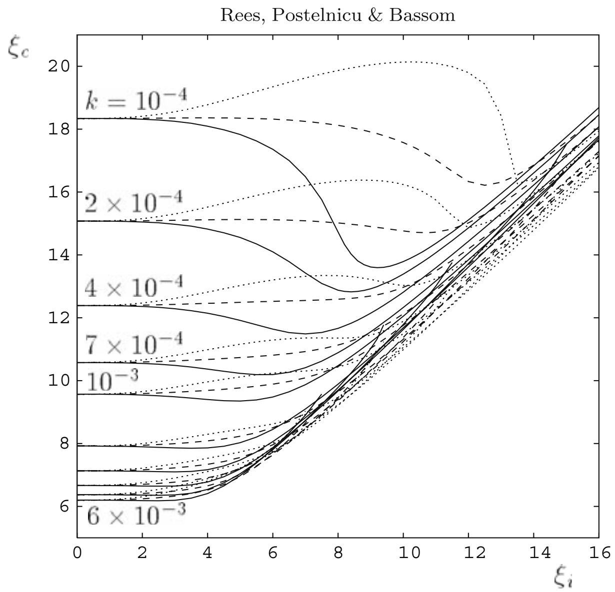


Fig. 8 Showing the effect on the neutral distance, ξ_c, of varying ξ_i and η_i where the disturbance profile $\Theta = e^{-(\eta-\eta_i)^2}$ is introduced at $\xi = \xi_i$. Continuous curves correspond to $\eta_i = 0$, dashed lines to $\eta_i = 1$ and dotted lines to $\eta_i = 2.393072$ (c.f. Eq. 5.10). The wavenumbers represented are given by $10^4 k = 1, 2, 4, 7, 10, 20, 30, 40, 50$ and 60

5 Asymptotic Behaviour of the Neutral Curve

This section is devoted to a brief description of the behaviour of the right hand branch of the neutral curve shown in Fig. 4. This is important because it becomes increasingly difficult to compute the right hand branch when k is very small due to the fact that the disturbance has a very large region of growth. Our aim here is to find an analytical formula for the shape of the branch.

A detailed examination of the numerical solutions indicate that not only are the vortex disturbances centred at a positive value of η (since this is where the plume is unstably stratified), but that they appear to narrow slightly as ξ increases. This latter observation is quite common for the right hand branch of neutral curves for developing boundary layers. We note that, at large values of ξ, the evolution of disturbances is such that all memory of the form and location of the initial disturbance has been lost.

The disturbance is assumed to be centred at $\eta = \eta_0$, a value which is to be determined below, and we find that it extends from this line by an $O(\xi^{-1/4})$ distance. We therefore take the following scalings:

$$\eta = \eta_0 + \xi^{-1/4} Y, \tag{5.1}$$

$$P(\xi, \eta) = \xi^{1/4} \sum_{n=0}^{\infty} \xi^{-n/4} P_n(Y), \tag{5.2}$$

$$\theta(\xi, \eta) = \sum_{n=0}^{\infty} \xi^{-n/4} \theta_n(Y), \tag{5.3}$$

$$k = \xi^{-3/2} \sum_{n=0}^{\infty} \xi^{-n/4} k_n. \tag{5.4}$$

In addition, we expand the basic state, f, which is given by Eq. 2.16, in a standard Taylor series about $\eta = \eta_0$:

$$f(\eta) = f_0 + \sum_{n=1}^{\infty} \xi^{-1/4} f_0^{(n)} \frac{Y^n}{n!}, \tag{5.5}$$

where $f_0 = f(\eta_0)$ and where $f_0^{(n)}$ is the value of the nth derivative of f at $\eta = \eta_0$. Our aim is to determine values for the k_n terms in (5.4). The above series are substituted into Eqs. 3.5 and coefficients of like powers of ξ are equated. At leading order (i.e., $O(\xi)$) in the expansion of (3.7b) we obtain,

$$k_0^2 \theta_0 = f_0^{(2)} \theta_0, \tag{5.6}$$

from which we infer that,

$$k_0 = \sqrt{-f_0^{(2)}}, \tag{5.7}$$

which will be evaluated later. At the next order we obtain,

$$-2k_0 k_1 \theta_0 = f_0^{(3)} Y \theta_0, \tag{5.8}$$

and therefore we must have both,

$$k_1 = 0, \quad \text{and} \quad f_0^{(3)} = 0. \tag{5.9}$$

This latter condition corresponds to where the temperature gradient within the basic state takes its largest value. Hence we obtain,

$$\eta_0 = \frac{3}{c}\ln(2+\sqrt{3}) = 2.393072, \tag{5.10}$$

which is the location of the centreline of the vortex disturbance. It may now be shown that,

$$k_0 = 2^{-1/2}3^{-3/4} = 0.310202. \tag{5.11}$$

At $O(\xi^{1/2})$ in the expansion of Eq. 3.7b and at leading order (i.e. at $O(\xi^{5/4})$) in the expansion of Eq. 3.7a we obtain the pair of equations,

$$\theta_0'' - \left[2k_0k_2 - \tfrac{1}{2}Y^2 f_0^{(4)}\right]\theta_0 = -f_0^{(2)}P_0', \quad k_0^2 P_0 = -\theta_0'. \tag{5.12}$$

Use of the results obtained so far, and the elimination of P_0 yields the equation,

$$\theta_0'' - \left[k_0k_2 + \tfrac{1}{4}f_0^{(4)}Y^2\right]\theta_0 = 0, \tag{5.13}$$

which is the parabolic cylinder equation. This equation admits eigensolutions for specific values of k_2; on scaling (5.13) into the standard form, it is straightforward to show that the leading eigenvalue and eigensolution are,

$$k_2 = -2^{-4/3}3^{-1/3}, \quad \theta_0 = e^{-\bar{Y}^2/2}, \tag{5.14}$$

where

$$\bar{Y} = (\tfrac{1}{4}f_0^{(4)})^{1/4}Y. \tag{5.15}$$

Further analysis rapidly becomes very lengthy to present, but proceeding in the same way it is possible to show, first, that $k_3 = 0$ at the next order of the expansion, and that

$$\theta_1 = 2^{-5/12}3^{-55/24}\left[\bar{Y}^3 - \bar{Y}\right]e^{-\bar{Y}^2/2}. \tag{5.16}$$

Finally, we obtain

$$k_4 = 0.341280, \tag{5.17}$$

and therefore the three-term approximation to the right hand branch of the neutral curve is

$$k \sim 0.310202\xi^{-3/2} - 0.275161\xi^{-2} + 0.341280\xi^{-5/2}, \tag{5.18}$$

which is shown in Fig. 4. We note that, when $k = 0.001$, the parabolic simulation yields $\xi = 40.804396$, while the three-term large-ξ approximation gives $\xi = 42.313615$, which is in error by less than 4%.

We mention that while it is possible to construct an asymptotic description of the right hand branch of the neutral curve, an equivalent calculation for the left hand branch is not feasible. This branch is described by a fully non-parallel theory, in the sense that its location is sensitive to both the initial disturbance profile imposed at ξ_i, and the value of ξ_i itself, as depicted in Fig. 8. This is very reminiscent of the situation that arises in the development of Görtler vortices in spatially developing curved boundary layers; it was first shown by Hall (1982) that the position of the neutral curve for this problem is a function of initial conditions. What we can infer from Fig. 4 is that for a vortex of wavenumber $k < 0.00833$, an imposed structure decays for some distance, then encounters the left hand branch of the neutral curve and subsequently grows. This growth is only halted once the right hand branch is reached and the location of this branch is virtually independent of the initial conditions.

6 Conclusions

In this article we have considered the stability characteristics of a plume which is generated by a line source that is embedded in a nearly horizontal surface. The inclination of the surface causes the centreline of the plume to be aligned at a small angle from the vertical, thereby rendering it susceptible to instability. Using the boundary layer approximation as the only approximation, the linearised stability equations become parabolic, and therefore we have monitored the evolution of vortex disturbances which were seeded near to the line source.

We have found that the neutral curve is independent of the shape (i.e. the cross-sectional profile) of the vortex and the position of its introduction whenever it is introduced upstream of $\xi = 3$, and whenever the wavenumber is not too small. The critical values of ξ and the wavenumber, k, are 6.014 and 0.00691, respectively.

In dimensional terms the critical distance may now be written in terms of ϵ, the inclination of the plane from the horizontal. Using the transformations (2.4a), (2.7a) and (3.3), and recalling that δ is given in terms of ϵ in (2.11), the critical distance is found easily to be,

$$\overline{x} = \left(\frac{7 \times 6.014}{4\epsilon}\right)^3 d \simeq 1166 d\epsilon^{-3}. \tag{6.19}$$

As this critical distance recedes to infinity as $\epsilon \to 0$, we may also conclude that the vertical plume is linearly stable to vortex disturbances. This conclusion is in qualitative agreement with the studies of Lewis et al. (1995) and Rees (1993) who showed that the vertical boundary layer which is driven by a uniformly hot surface is also stable.

We have also found that the evolution of disturbances with relatively small wavenumbers is very sensitive to the initial disturbance profile and the position at which the disturbance has been introduced into the plume. Even disturbances which are initiated at very small values of ξ_i do not necessarily provide the smallest value of ξ_c, but in those cases where they do not, the wavenumber of the vortex is substantially below the value given in Eq. 4.1, which remains the smallest value of ξ_c for the system.

Acknowledgements This article was completed while the first author (DASR) was enjoying study leave at the University of Bristol. He is grateful to his hosts in the Department of Mathematics for their hospitality.

References

Afzal, N.: Two-dimensional buoyant plume in porous media: higher order effects. Int. J. Heat Mass Transf. **28**, 2029–2041 (1985)

Bassom, A.P., Rees, D.A.S., Storesletten, L.: Convective plumes in porous media: the effect of asymmetrically placed boundaries. Int. Comm. Heat Mass Transf. **28**, 31–38 (2001)

Hall, P.: Taylor-Görtler vortices in fully developed or boundary-layer flows: linear theory. J. Fluid Mech. **124**, 475–494 (1982)

Ingham, D.B.: An exact solution for non-Darcy free convection from a horizontal line source. Wärme-Stoffübertrag. **22**, 125–127 (1988)

Lewis, S., Bassom, A.P., Rees, D.A.S.: The stability of vertical thermal boundary layer flow in a porous medium. Eur. J. Mech. B-Fluids **14**, 395–407 (1995)

Nield, D.A., Bejan, A.: Convection in Porous Media, 3rd edn. Springer, New York (2006)

Rees, D.A.S.: A numerical investigation of the nonlinear wave stability of vertical thermal boundary layer flow in a porous medium. Z.A.M.P. **44**, 306–313 (1993)

Rees, D.A.S.: Vortex instability from a near-vertical heated surface in a porous medium. I Linear theory. Proc. R. Soc. A **457**, 1721–1734 (2001)

Rees, D.A.S., Hossain, M.A.: The effect of inertia on free convective plumes in porous media. Int. Comm. Heat Mass Transf. **28**, 1137–1142 (2001)

Rees, D.A.S., Storesletten, L., Bassom, A.P.: Convective plume paths in anisotropic porous media. Transp. Porous Media **49**, 9–25 (2002)

Wooding, R.A.: Convection in a saturated porous medium at large Reynolds number of Péclet number. J. Fluid Mech. **9**, 183–192 (1963)