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International Journal of Thermal Sciences 47 (2008) 1123–1130

International Journal of Thermal Sciences

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# The Entrainment Theorem for the Darcy free convection over a permeable vertical plate

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Received 29 August 2007; received in revised form 20 October 2007; accepted 23 October 2007

Available online 26 November 2007

**Abstract**

The Cheng–Minkowycz model of the Darcy free convection boundary layer flow over a permeable vertical plate with prescribed power-law temperature distribution $T_w(x) = T_\infty + A \cdot x^\lambda$ and an applied lateral mass flux is revisited in this paper. The relationship between the wall heat flux and the entrainment velocity (the similar transversal velocity at the outer edge of the boundary layer) as function of the mass transfer parameter $f_w$ is examined analytically by using the Merkin transformation method. It is shown that at the value of $f_w$ where the Nusselt number becomes zero and changes sign, the entrainment velocity passes through its minimum value (*Entrainment Theorem*). The converse statement is also true, and holds for all the surface temperature distributions with power-law exponent in the range $-1 < \lambda < 0$. It also applies to the Darcy free convection over a permeable vertical plate with exponential temperature distribution when the effect of viscous dissipation is significant.


*Keywords:* Porous medium; Free convection; Boundary layer; Nusselt number; Entrainment velocity; Merkin transformation

## 1. Introduction

Following the seminal work of Cheng and Minkowycz [1] and Cheng [2,3], the theory of self-similar boundary layer flows in fluid saturated porous media has experienced a rapid development. A comprehensive review of this development of broad theoretical and practical interest can be found in the monographs of Pop and Ingham [4] and Nield and Bejan [5], for example. Extensions to the non-similar case of constant surface temperature and constant transpiration velocity were reported by Merkin [6] and Minkowycz [7].

The present paper revisits the Cheng–Minkowycz model [1,2] of the Darcy free convection boundary layer flow over a permeable vertical plate with prescribed power-law temperature distribution $T_w(x) = T_\infty + A \cdot x^\lambda$, $A > 0$, and an applied lateral mass flux proportional to $x^{(\lambda-1)/2}$. Its focus is on a specific physical and mathematical aspect of this model, namely on the relationship between the wall heat flux and the entrainment velocity as function of the mass transfer parameter $f_w$ (suction/injection parameter). The main result of the paper is summarized in a short statement referred to as *entrainment theorem,* which asserts that at the value of the mass transfer parameter where the wall heat flux changes sign, the entrainment velocity of the flow passes through its minimum value. The *entrainment theorem* holds for all $-1 < \lambda < 0$, and its converse is also true. It also applies to the Darcy free convection over a permeable vertical plate with exponential temperature distribution when the effect of viscous dissipation is significant.

## 2. Basic equations and problem formulation

We consider the Cheng–Minkowycz model [1,2] of the self-similar Darcy free convection boundary layer flow over a permeable vertical plate with prescribed power-law surface temperature distribution $T_w(x) = T_\infty + A \cdot x^\lambda$. Hereafter the notations of [1] and [2] will be used. Following Chaudhary et al. [8], for the wall temperature exponent $\lambda$ the variation range $\lambda > -1$ will be admitted.

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1290-0729/$ – see front matter 
doi:10.1016/j.ijthermalsci.2007.10.008

**Nomenclature**

| | |
|---|---|
| $A$ | wall temperature coefficient, Eq. (7) |
| $A_n$ | dimensionless expansion coefficients, Eq. (18) |
| $f$ | dimensionless stream function |
| $f_w$ | mass transfer parameter, Eq. (3) |
| $f_\infty$ | similar entrainment velocity, Eq. (7) |
| $H$ | dimensionless heat transfer parameter, Eq. (6) |
| $T_w$ | wall temperature distribution, $= T_\infty + Ax^\lambda$ |
| $T_\infty$ | ambient temperature |
| $x$ | dimensional wall coordinate |
| $y$ | dimensional transversal coordinate |
| $Y$ | dimensionless function, Eq. (12) |
| $z$ | dimensionless variable, Eq. (12) |
| $z_0$ | $z$ at the wall, Eq. (15) |

*Greek symbols*

| | |
|---|---|
| $\beta$ | parameter, Eq. (15) |
| $\lambda$ | wall temperature exponent |
| $\eta$ | similarity independent variable, Eq. (5) |
| $\theta$ | dimensionless temperature, Eq. (5) |

*Subscripts*

| | |
|---|---|
| $w$ | values at the wall |
| $\infty$ | values at infinity (outer edge of the boundary layers) |
| $k, n$ | summation indices |

*Superscripts*

| | |
|---|---|
| dashes | derivatives with respect to $\eta$ |
| dots | derivatives with respect to $f_w$ |

The self-similar stream function $f = f(\eta)$ and temperature $\theta = \theta(\eta)$ fields are obtained as solutions of the two point boundary value problem [1,2]

$$f'' - \theta' = 0 \tag{1}$$

$$\theta'' + \frac{1+\lambda}{2} f\theta' - \lambda f'\theta = 0 \tag{2}$$

$$f(0) = f_w, \quad \theta(0) = 1 \tag{3}$$

$$f'(\infty) = 0, \quad \theta(\infty) = 0 \tag{4}$$

where the prime denotes differentiation with respect to the similarity variable $\eta$, and $f_w$ stands for the mass transfer parameter (or similar suction/injection velocity; with $f_w > 0$ corresponding to suction, and $f_w < 0$ to injection). On integrating Eq. (1) and applying the boundary conditions (4) one immediately determines that the temperature $\theta(\eta)$ is identical to the self-similar streamwise velocity $f'(\eta)$ of the flow, [1,2],

$$\theta(\eta) = f'(\eta) \tag{5}$$

Thus, all the quantities of physical and engineering interest can be calculated in terms of the self-similar stream function $f = f(\eta)$. The heat transferred through the wall is characterized by the dimensionless group

$$\frac{Nu_x}{\sqrt{Ra_x}} = -\theta'(0) = -f''(0) \equiv H \tag{6}$$

where $Nu_x$ and $Ra_x$ are the local Nusselt and Darcy–Rayleigh numbers, respectively, [1,2]. For simplicity, the dimensionless group $H \equiv Nu_x/\sqrt{Ra_x}$ will be named the *heat transfer parameter*.

The dimensional entrainment velocity $v(x,\infty)$ is the transversal component $v(x,y)$ of the velocity field $\mathbf{v}(u(x,y), v(x,y), 0)$ at the outer edge of the boundary layer, [1,2],

$$v(x,\infty) = -(1+\lambda)\big[\alpha\rho_\infty g\beta K A x^{\lambda-1}/(4\mu)\big]^{1/2} f(\infty) \tag{7}$$

where $f(\infty) \equiv f_\infty$ will be referred to as *similar entrainment velocity*.

The aim of the present paper is to investigate the relationship between the heat transfer parameter $H = H(\lambda; f_w)$ and the similar entrainment velocity $f_\infty = f_\infty(\lambda; f_w)$ in the parameter plane $(\lambda, f_w)$ specified by the wall temperature exponent $\lambda$ and the mass transfer parameter $f_w$. According to our knowledge, this problem has not yet been examined in detail. Concerning the effect of $f_w$ on $H = H(\lambda; f_w)$ and $f_\infty = f_\infty(\lambda; f_w)$, the following theorem will be proven below.

*At the value $f_w = f_w^*$ of the mass transfer parameter where the wall heat flux vanishes ($H(\lambda; f_w^*) = 0$) and changes sign, the similar entrainment velocity of the flow $f_\infty = f_\infty(\lambda; f_w)$ passes through its minimum value $f_{\infty,\min} = f_\infty(\lambda; f_w^*)$. The converse statement is true, too.*

This theorem holds for all $\lambda$ values in the interval $-1 < \lambda < 0$ and will be referred to in shortened form as the *entrainment theorem*. The *entrainment theorem* is the main result of the paper.

## 3. Analytical solutions

### 3.1. The special cases $\lambda = -1/3$ and $\lambda = -1/2$

As it is well known, [9,10], in the special cases $\lambda = -1/3$ and $\lambda = -1/2$ the solution of the boundary value problem (1)–(4) can be given in terms of elementary transcendental functions.

- Case $\lambda = -1/3$, [9,10]

$$H = \frac{1}{3} f_w, \quad -\infty < f_w < +\infty$$

$$f_\infty = \sqrt{f_w^2 + 6} = \sqrt{9H^2 + 6}$$

$$f(\eta) = f_\infty \tanh\left(\frac{1}{6} f_\infty \eta + \ln\sqrt{\frac{f_\infty + f_w}{f_\infty - f_w}}\right) \tag{8}$$

- Case $\lambda = -1/2$, [9]

$$H = \frac{1}{4}\left(f_w - \frac{2}{f_w}\right), \quad 0 < f_w < +\infty$$

$$f_\infty = \left(f_w^3 + 12 f_w + \frac{36}{f_w}\right)^{1/3} \tag{9}$$

The solution for the similar stream function $f = f(\eta)$ can be given for $\lambda = -1/2$ only in the implicit form $\eta = \eta(f)$ (for further details see [9]). It is worth emphasizing here that, while the solution (8) exists for all values $-\infty < f_w < +\infty$ of the mass transfer parameter, the implicit solution $\eta = \eta(f)$ corresponding to $\lambda = -1/2$ exists only when a lateral suction ($0 < f_w < +\infty$) is applied, [9].

### 3.2. The Merkin transformation

To prove the *entrainment theorem* for all $-1 < \lambda < 0$, an extension of the Merkin transformation approach [11] to the present boundary value problem (1)–(4) will be used. Eliminating the dimensionless temperature $\theta$ with the aid of Eq. (5), we first transcribe the boundary value problem (1)–(4) in the form

$$f''' + \frac{1+\lambda}{2} f f'' - \lambda f'^2 = 0 \tag{10}$$

$$f(0) = f_w, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{11}$$

The basic feature of the Merkin transformation is that it reverses the role of the stream function $f$ in the boundary value problem (10), (11) from that of the old dependent variable to that of a new independent variable $\phi \equiv f_\infty - f$ and at the same time, it transfers the role of the dependent variable from $f$ to $p(\phi) \equiv \mathrm{d}f/\mathrm{d}\eta$. The main advantage of this transformation is that it enables the calculation of the heat transfer parameter $H = H(\lambda; f_w)$ and the entrainment velocity $f_\infty = f_\infty(\lambda; f_w)$ without needing to know the solution $f = f(\eta; \lambda, f_w)$ of the boundary value problem (10), (11), neither in an explicit, nor in an implicit form.

Firstly, we extend the Merkin transformation method [11], developed originally for an impermeable surface $f_w = 0$, to the case $f_w \neq 0$ in which a lateral mass flux is present. To this end, we modify the transformation slightly by changing to a new independent variable $z$ and to a new dependent one, $Y = Y(z)$, which we define as follows:

$$z = 1 - \frac{f}{f_\infty}, \qquad Y = \frac{2}{(1+\lambda) f_\infty^2} \frac{\mathrm{d}f}{\mathrm{d}\eta} \tag{12}$$

Thus, the boundary value problem (10), (11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(Y \frac{\mathrm{d}Y}{\mathrm{d}z}\right) + (z-1)\frac{\mathrm{d}Y}{\mathrm{d}z} - \beta Y = 0 \tag{13}$$

$$Y(0) = 0, \qquad Y(z_0) = \frac{2}{(1+\lambda) f_\infty^2} \tag{14}$$

where

$$z_0 = 1 - \frac{f_w}{f_\infty}, \qquad \beta = \frac{2\lambda}{1+\lambda} \tag{15}$$

The first condition (14) has been obtained from $f'(\infty) = 0$ and the second one from $f(0) = f_w$ and $f'(0) = 1$. We mention that, in the case $\beta = 2$ (obtained formally from Eq. (15) for $\lambda \to \infty$), the boundary value problem (10), (11) describes the Darcy free convection over the vertical plate with exponential temperature distribution, when the effect of viscous dissipation is significant, [12,13].

The heat transfer parameter $H$ is obtained in this approach as

$$H = \frac{1}{2}(1+\lambda) f_\infty \left.\frac{\mathrm{d}Y}{\mathrm{d}z}\right|_{z=z_0} \tag{16}$$

After the solution $Y = Y(z)$ of the boundary value problem (13), (14) has been found, the solution $f = f(\eta)$ of the original problem (10), (11) can be obtained in the implicit form $\eta = \eta(f)$ by quadratures,

$$\eta = -\frac{2}{(1+\lambda) f_\infty} \int_{z_0}^{1-\frac{f}{f_\infty}} \frac{\mathrm{d}z}{Y(z)} \tag{17}$$

### 3.3. The series solution

Looking for the solution of the boundary value problem (13), (14) in the power series form

$$Y = \sum_{n=0}^{\infty} A_n z^n \tag{18}$$

one obtains for the coefficients $A_n$ the system of equations, [14],

$$\sum_{n=0}^{k} (n+1)\big[(n+2) A_{n+2} A_{k-n} + (k-n+1) A_{n+1} A_{k-n+1}\big] = (k+1) A_{k+1} + (\beta - k) A_k, \quad k = 0, 1, 2, \ldots \tag{19}$$

The boundary condition $Y(0) = 0$ implies that $A_0 = 0$. Thus, one obtains from Eq. (19) the following expressions for the next two coefficients

$$A_1 = 1, \qquad A_2 = \frac{1}{4}(\beta - 1) \tag{20}$$

The subsequent coefficients $A_3, A_4, A_5, \ldots$ can then be obtained recursively according to

$$A_k = \frac{\beta - k + 1}{k^2} A_{k-1} - \frac{k+1}{2k} \cdot \sum_{n=2}^{k-1} A_n A_{k-n+1}, \quad k = 3, 4, 5, \ldots \tag{21}$$

Specifically, we have

$$\begin{aligned} A_3 &= \frac{1}{72}(1-\beta^2) \\ A_4 &= \frac{1}{576}(1-\beta^2)(1-2\beta) \\ A_5 &= \frac{1}{86400}(1-\beta^2)(11 - 81\beta + 88\beta^2) \\ A_6 &= \frac{1}{1036800}(1-\beta^2)(-9 - 125\beta + 447\beta^2 - 337\beta^3) \end{aligned} \tag{22}$$

Then, the second boundary condition (14) yields for $f_\infty$ the equation

$$\frac{1+\lambda}{2} f_\infty^2 \sum_{k=1}^{\infty} A_k \left(1 - \frac{f_w}{f_\infty}\right)^k - 1 = 0 \tag{23}$$

For the heat transfer parameter $H = H(\lambda; f_w)$ one obtains from Eq. (16) the expression

$$H = \frac{1}{2}(1+\lambda) f_\infty \sum_{k=1}^{\infty} k A_k \left(1 - \frac{f_w}{f_\infty}\right)^{k-1} \tag{24}$$

## 4. Discussion

### 4.1. The proof

A first analytical proof of the *entrainment theorem* can be given with the aid of Eqs. (23) and (24). Indeed, on differentiating Eq. (23) once with respect to $f_w$ one obtains

$$2 f_\infty \frac{\mathrm{d}f_\infty}{\mathrm{d}f_w} \sum_{k=1}^{\infty} A_k \left(1 - \frac{f_w}{f_\infty}\right)^{k} - \left(f_\infty - f_w \frac{\mathrm{d}f_\infty}{\mathrm{d}f_w}\right) \sum_{k=1}^{\infty} k A_k \left(1 - \frac{f_w}{f_\infty}\right)^{k-1} = 0 \tag{25}$$

On solving this equation with respect to $\mathrm{d}f_\infty/\mathrm{d}f_w$ and taking into account Eqs. (23) and (24) again, one obtains the relationship

$$\frac{\mathrm{d}f_\infty}{\mathrm{d}f_w} = \frac{f_\infty H}{f_w H + 2} \tag{26}$$

This equation shows that, when at some value $f_w = f_w^*$ of the mass transfer parameter the heat transfer parameter $H = H(\lambda; f_w)$ possesses (for a specified $\lambda$) a zero, then, $f_w = f_w^*$ yields at the same time a zero of the derivative $\mathrm{d}f_\infty/\mathrm{d}f_w$ of the entrainment velocity, and conversely. This conclusion is in agreement with the statement of the *entrainment theorem*. However it leaves the question open, whether the common root $f_w = f_w^*$ of equations $H = 0$ and $\mathrm{d}f_\infty/\mathrm{d}f_w = 0$ corresponds to a minimum, a maximum or to an inflexion point of the function $f_\infty = f_\infty(\lambda; f_w)$. A more general and detailed analytical proof of the *entrainment theorem*, supported directly by the basic equations (10), (11) of our two point boundary value problem, is given in Appendix A.

### 4.2. *The special cases* $\lambda = -1/3$ *and* $\lambda = -1/2$

The special solutions given in Section 3.1 offer a good opportunity for a simple straightforward validation of the *entrainment theorem*.

In the case $\lambda = -1/3$, the statement of this theorem becomes evident by a simple inspection of Eqs. (8). Indeed, according to the first equations (8), the heat transfer parameter $H$ becomes zero for $f_w \equiv f_w^* = 0$ (impermeable surface), where, according to the second equation (8), the entrainment velocity $f_\infty$ actually reaches its minimum value, $f_{\infty,\min} = \sqrt{6}$. In the case $\lambda = -1/2$, elementary calculations show that both the heat transfer parameter $H$ and the first derivative of $f_\infty$ with respect to $f_w$, $\mathrm{d}f_\infty/\mathrm{d}f_w = (f_w^2 - 2)(f_w^2 + 6)/(f_w f_\infty)^2$, become zero for $f_w = \sqrt{2}$. Thus $f_w^* = \sqrt{2}$, and the corresponding minimum value of the entrainment velocity is $f_{\infty,\min} = 2^{11/6} = 3.5636$. These two particular cases of the *entrainment*

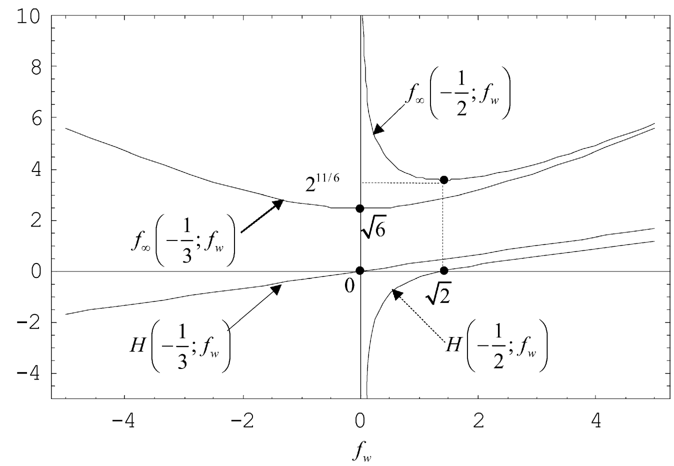


Fig. 1. Plots of the similar entrainment velocity $f_\infty = f_\infty(\lambda; f_w)$ and the heat transfer parameter $H = H(\lambda; f_w)$ as functions of the mass transfer parameter $f_w$ for $\lambda = -1/3$ and $\lambda = -1/2$, respectively.

*theorem* are illustrated in Fig. 1, where the respective quantities $f_\infty = f_\infty(\lambda; f_w)$ and $H = H(\lambda; f_w)$ have been plotted as functions of $f_w$. One sees that $f_\infty(\lambda; f_w^*)$, with $f_w^* = 0$ for $\lambda = -1/3$, and $f_w^* = \sqrt{2}$ for $\lambda = -1/2$, actually is a minimum of the entrainment velocity. Thus, according to the *entrainment theorem,* $H(\lambda; f_w^*) = 0$, such that the wall heat flux changes sign at $f_w = f_w^*$. In the case $\lambda = -1/2$, for example, the heat is transferred from the wall to the fluid (a direct heat flux) only in the range $f_w > f_w^* = \sqrt{2}$, while for $0 < f_w < f_w^* = \sqrt{2}$, the heat transfer takes place from the fluid to the wall (reversed heat flux). At $f_w = f_w^* = \sqrt{2}$ where $f_\infty = f_{\infty,\min}$, the wall is adiabatic, and $H(\lambda; \sqrt{2}) = 0$. In the case $\lambda = -1/3$, the sign change of the wall heat flux happens at $f_w = f_w^* = 0$. Furthermore, it is an elementary exercise to show that in the case of the special solutions (8) and (9), the derivative $\mathrm{d}f_\infty/\mathrm{d}f_w$ can also be put into the generally valid form (26).

### 4.3. *The full interval* $-1 < \lambda < 0$

In the general case $-1 < \lambda < 0$, the results of the numerical investigation of the *entrainment theorem* based on Eqs. (23) and (24) are illustrated graphically in Fig. 2 for five selected values of the temperature exponent, namely: $\lambda = -0.75, -0.65, -0.50, -0.45, -1/3, -0.20$. The family of entrainment velocity curves $f_\infty = f_\infty(\lambda; f_w)$ represents the contour plots (topographic maps) of Eq. (23) for the selected values of $\lambda$. The family of the corresponding heat transfer curves $H = H(\lambda; f_w)$ must then be obtained by substituting the explicit (numerical) solution of Eq. (23) into Eq. (24), and plotting the resulting expressions of $H$ as functions of the mass transfer parameter $f_w$. Numerically, we find that, for any specified value of $\lambda$ in the interval $-1 < \lambda < 0$, the minima $f_{\infty,\min} = f_\infty(\lambda; f_w^*)$ of the entrainment velocity on the one hand, and the vanishing values $H(\lambda; f_w^*)$ of the heat transfer parameter on the other hand, are reached at the same values $f_w^*(\lambda)$ of the mass transfer parameter $f_w$, in full agreement with the *entrainment theorem.* The corresponding points of the two families of curves are marked in Fig. 2 by dots, and the respective numerical val-

Table 1
Values $f_w^*$ of the mass transfer parameter $f_w$ for which the similar entrainment velocity $f_\infty(\lambda; f_w)$ reaches (in the interval $-1 < \lambda < 0$) its minimum $f_{\infty,\min} = f_\infty(\lambda; f_w^*)$ and the heat transfer parameter vanishes, $H(\lambda; f_w^*) = 0$

| $\lambda$ | $f_w^*$ | $f_{\infty,\min}$ | $\lambda$ | $f_w^*$ | $f_{\infty,\min}$ |
|---|---|---|---|---|---|
| −0.99 | 213.468 | 215.145 | −0.475 | 1.16656 | 3.35329 |
| −0.98 | 105.547 | 107.231 | −0.45 | 0.93521 | 3.16153 |
| −0.95 | 40.7753 | 42.4793 | −0.40 | 0.51093 | 2.82420 |
| −0.925 | 26.3663 | 28.0878 | −1/3 | 0 | $\sqrt{6}$ |
| −0.90 | 19.1508 | 20.8905 | −0.30 | −0.24173 | 2.28717 |
| −0.875 | 14.8124 | 16.5708 | −0.25 | −0.59877 | 2.06842 |
| −0.85 | 11.9121 | 13.6899 | −0.20 | −0.96336 | 1.87389 |
| −0.825 | 9.83326 | 11.6311 | −0.15 | −1.35787 | 1.69859 |
| −0.80 | 8.26748 | 10.0861 | −0.10 | −1.82423 | 1.53817 |
| −0.75 | 6.05820 | 7.92085 | −0.05 | −2.47976 | 1.38807 |
| −0.70 | 4.56445 | 6.47469 | $-10^{-2}$ | −3.64256 | 1.27009 |
| −0.65 | 3.47728 | 5.43922 | $-10^{-3}$ | −4.85551 | 1.24194 |
| −0.60 | 2.64188 | 4.66032 | $-10^{-4}$ | −5.79068 | 1.23885 |
| −0.55 | 1.97170 | 4.05225 | $-10^{-5}$ | −6.57006 | 1.23853 |
| −1/2 | $\sqrt{2}$ | $2^{11/6}$ | $-10^{-6}$ | −7.25949 | 1.23898 |

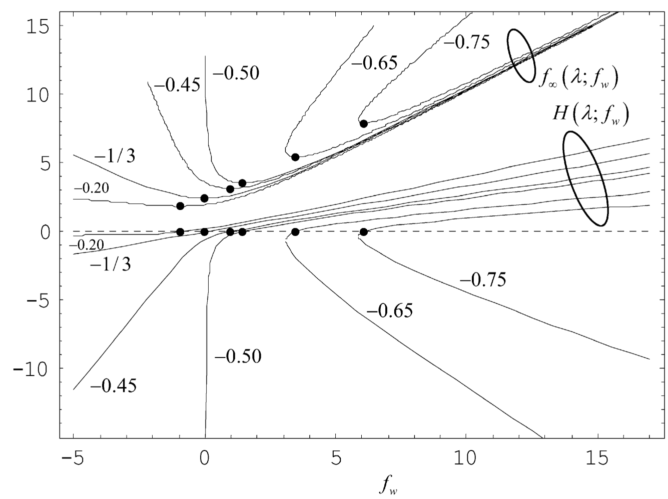


Fig. 2. Plots of the entrainment velocities $f_\infty = f_\infty(\lambda; f_w)$ as functions of the mass transfer parameter $f_w$ for six different values of the power-law exponent $\lambda$ in the range $-1 < \lambda < 0$. The minima of the curves $f_\infty(\lambda; f_w)$, marked by dots, are associated with *adiabatic temperature profiles*, i.e. with flows of vanishing heat transfer parameter, $H(\lambda; f_w^*) = 0$, in full agreement with the *entrainment theorem.*

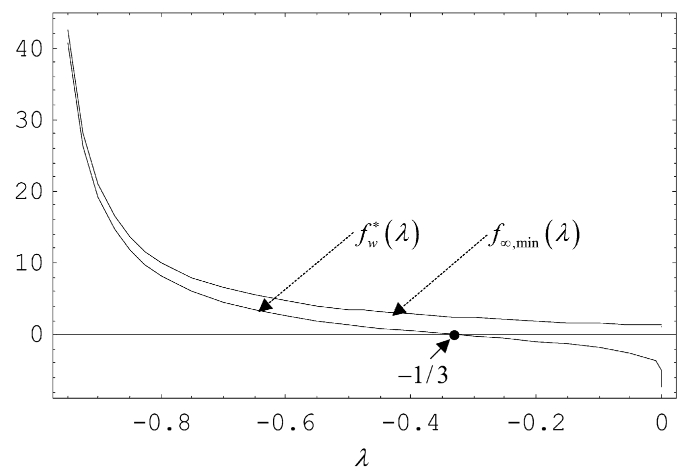


Fig. 3. Dependence on the temperature exponent $\lambda$, $-1 < \lambda < 0$, of the minimum entrainment velocity $f_{\infty,\min}(\lambda)$ and of the corresponding value $f_w^*(\lambda)$ of the mass transfer parameter $f_w$.

ues of $f_w^*$ and $f_{\infty,\min}$, along with those obtained for further 24 values of $\lambda$ in the interval $-1 < \lambda < 0$, are collected in Table 1.

A further important consequence of Eqs. (23) and (24), is that, for finite values of $f_w (-\infty < f_w < +\infty)$, no zeros of the heat transfer parameter $H(\lambda; f_w)$ can exist when $\lambda \geqslant 0$. In the range $\lambda \geqslant 0$, the boundary value problem (1)–(4) admits solutions only for $H(\lambda; f_w) > 0$.

The elementary analytical results (8) and (9) corresponding to the values $\lambda = -1/3$ and $\lambda = -1/2$ could also be recovered from Eqs. (23) and (24) exactly, and have been included in Fig. 2 and Table 1, too. For sake of transparency, the values of $f_w^*$ and $f_{\infty,\min}$ included in Table 1 have been plotted as functions of $\lambda$ in Fig. 3. This figure shows that both $f_w^*$ and $f_{\infty,\min}$ decrease monotonically as $\lambda$ increases from $-1$ toward 0. However, while $f_{\infty,\min}$ is positive for all $-1 < \lambda < 0$, $f_w^*$ changes sign at $\lambda = -1/3$, becoming negative in the range $-1/3 < \lambda < 0$. This feature can also be proven analytically with the aid of the integral relationship

$$H = \frac{1+\lambda}{2} f_w + \frac{1+3\lambda}{2} \cdot \int_0^\infty f'^2(\eta)\,\mathrm{d}\eta \tag{27}$$

which is a straightforward consequence of Eqs. (10) and (11). Indeed, for $H(\lambda; f_w^*) = 0$ and $-1 < \lambda < 0$, Eq. (27) implies

$$\operatorname{sgn} f_w^* = -\operatorname{sgn}(1+3\lambda) \tag{28}$$

which clearly shows that $f_w^* > 0$ for $-1 < \lambda < -1/3$ and $f_w^* < 0$ for $-1/3 < \lambda < 0$, in full agreement with Fig. 3 and Table 1.

Furthermore, the *adiabatic temperature profiles*, i.e. the solutions $\theta = \theta(\eta) = f'(\eta)$ corresponding to the vanishing values $H(\lambda; f_w^*) = 0$ of the heat transfer parameter $H$ predicted by the entrainment theorem, are also of physical interest. Based on the data of Table 1, in Fig. 4 such profiles are shown for five selected values of the power law exponent $\lambda$. It is seen that the

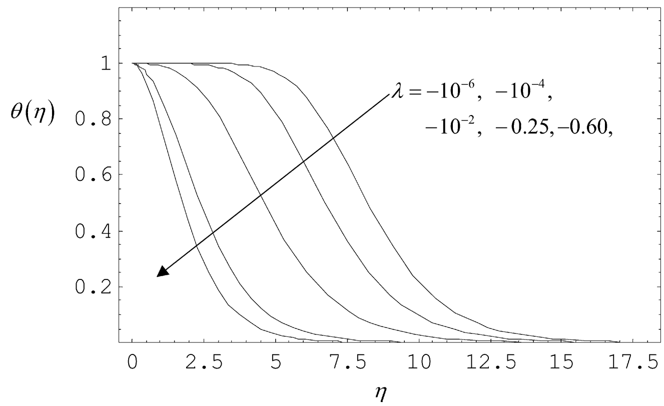


Fig. 4. Adiabatic temperature profiles corresponding to the indicated values of temperature exponent $\lambda$.

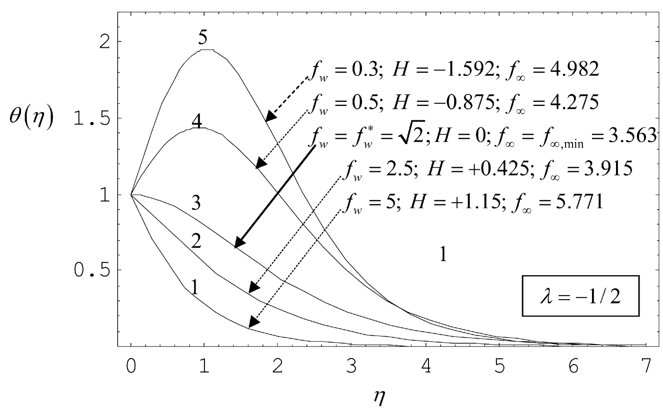


Fig. 5. Five dimensionless temperature profiles for $\lambda = -1/2$. The middle one represents the adiabatic profile with $(H, f_\infty) = (0, f_{\infty,\min})$, which corresponds to $f_w = f_w^* = \sqrt{2}$. The upper two profiles with $f_w < f_w^*$, are associated with negative values of $H$ (reversed wall heat flux) and the lower ones with $f_w > f_w^*$, correspond to positive values of $H$ (direct wall heat flux).

thickness of the adiabatic temperature boundary layers decrease as the values of $\lambda$ decrease toward the value $-1$.

The physical origin of the *entrainment theorem* has to be sought in the sign change of the wall heat flux. This phenomenon is illustrated in Fig. 5, where the temperature profiles corresponding to five selected values of the mass transfer parameter $f_w$ have been plotted as functions of $\eta$ for $\lambda = -1/2$. One of them (profile 3) is the *adiabatic temperature profile* corresponding to $H = H(\lambda; f_w^*) = 0$ and $f_\infty = f_{\infty,\min} = f_\infty(\lambda; f_w^*) = 2^{11/6}$ with $f_w^* = \sqrt{2}$. The other four profiles correspond to the indicated values of $f_w$, below of $f_w = f_w^* = \sqrt{2}$ (profiles 1 and 2) and above of $f_w = f_w^* = \sqrt{2}$ (profiles 4 and 5), and are associated with direct ($H > 0$) and reversed ($H < 0$) wall heat fluxes, respectively. Bearing in mind that the wall temperature $T_w(x) = T_\infty + A \cdot x^{-1/2}$, $A > 0$, is larger than the ambient temperature $T_\infty$ at all stations $x \geqslant 0$, the positive sign of wall heat fluxes associated with the temperature profiles 1 and 2 corresponds to our physical expectation. It is also clear in this case, that the larger the value of the suction parameter $f_w$, the larger is the corresponding entrainment velocity $f_\infty$ and heat transfer parameter $H$. This behaviour is encountered also for $\lambda \geqslant 0$. However, in the case of temperature profiles 4 and 5, it is not immediately clear, why the wall heat flux has changed its sign from $H > 0$ to $H < 0$, and why the entrainment velocities become larger than $f_{\infty,\min}$, although in this case $T_w(x)$ is still larger than $T_\infty$ for all $x \geqslant 0$ and the respective values of the suction parameter $f_w$ are *smaller* than $f_w^*$.

Obviously, the reversed wall heat fluxes (negative $H$'s) are related to the "temperature hills" of the $\theta$-profiles 4 and 5. But, where do these temperature peaks come from? Bearing in mind that the dimensionless temperature $\theta(\eta)$ coincides (see Eq. (5)) with the dimensionless streamwise velocity $f'(\eta)$, the temperature peaks which overshoot the local wall temperatures $T_w(x)$, represent at the same time also "velocity peaks". Since $\lambda < 0$, the velocity peaks transport *hot* fluid from upstream stations (small values of $x$) to all the downstream stations (larger values of $x$) of the flow. Consequently the heat flows from the "hot spots" into the wall (reversed heat flux) and, at the same time the velocity peaks lead in this case to values of the entrainment velocity which are larger than $f_{\infty,\min}$. The smaller the suction parameter, the higher the velocity peaks and the larger the corresponding entrainment velocities $f_\infty$ in the low suction range $0 < f_w < f_w^*$. In the strong suction range $f_w > f_w^*$ of the profiles 1 and 2 on the other hand, the velocity and temperature peaks are "sucked out" from the fluid and, consequently, the above-described "usual" heat transfer and entrainment mechanisms become effective. The latter ones persist then for all $\lambda \geqslant 0$.

It is worth mentioning here that a similar phenomenon where the wall heat flux changes sign, has already been noticed by Sparrow and Gregg [15] a half century ago in connection with the free convection of clear fluids over an impermeable vertical plate with a power law temperature distribution in the range $\lambda < 0$ of the temperature exponent. The occurrence of such "Sparrow–Gregg temperature hills" in heat transfer problems in wedge-type flows, has been discussed by Eckert and Drake [16] in some detail.

## 5. Summary and conclusions

In the present paper the Darcy free convection boundary layer flow over a permeable vertical plate with prescribed power-law temperature distribution and an applied lateral mass flux has been revisited. The relationship between the similar entrainment velocity and the wall heat flux has been investigated in detail. The main result of the paper is summarized in the *entrainment theorem* which asserts that, at the value of the mass transfer parameter where the wall heat flux becomes zero and changes its sign, the entrainment velocity of the flow passes through its minimum value. The *entrainment theorem* holds for all negative values of the wall temperature exponent and its converse is also true. This theorem represents at the same time a bridge between a flow characteristic at the outer edge and a heat transfer characteristic at the inner edge of the Darcy free convection boundary layer. The physical reason for this relationship resides in the temperature and velocity overshoot occurring for negative values of the wall temperature exponent (the "Sparrow–Gregg phenomenon"). The present authors believe that the *entrainment theorem* holds also in the case of free convection of clear fluids over a permeable vertical plate. The rigorous proof of this conjecture is still an open research opportunity.

**Acknowledgement**

The second author (DASR) would like to thank the University of Bristol for hosting his period of study leave, and to the Department of Mathematics for their hospitality.

## Appendix A

In the following, a straightforward general analytical proof of the *entrainment theorem* is given on the ground of Eqs. (10), (11) of our basic boundary value problem.

For convenience, we first introduce the notations

$$\frac{\partial f}{\partial f_w} = \dot{f}, \qquad \frac{\partial^2 f}{\partial f_w^2} = \ddot{f} \tag{A.1}$$

for the derivatives $\partial f/\partial f_w$ and $\partial^2 f/\partial f_w^2$ of $f = f(\eta)$ with respect to $f_w$. The proof will be accomplished by finding solutions for $\dot{f}(\eta)$and$\ddot{f}(\eta)$.

### A.1. First order analysis

The equation for $\dot{f}(\eta)$ is obtained by differentiating Eq. (10) with respect to $f_w$:

$$\dot{f}''' + \frac{1+\lambda}{2}(\dot{f} f'' + f \dot{f}'') - 2\lambda f' \dot{f}' = 0 \tag{A.2}$$

Its boundary conditions have to be chosen with care, and they are

$$\dot{f}(0) = 1, \qquad \dot{f}'(0) = 0, \qquad \dot{f}(\infty) = 0 \tag{A.3a,b,c}$$

Eqs. (A.3a,b) are the first $f_w$-derivatives of the equations $f(0) = f_w$ and $f'(0) = 1$, respectively, and Eq. (A.3c) is the expression of the *entrainment theorem* which states that $\dot{f}(\infty) = 0$ for the root $f_w \equiv f_w^*$ of $f''(0) = 0$.

Eq. (A.2) subject to (A.3) has the simple solution,

$$\dot{f}(\eta) = f'(\eta) \tag{A.4}$$

which, along with Eqs. (10) and (11) implies

$$\dot{f}''(0) = f'''(0) = \lambda \tag{A.5}$$

Now, we expand $f(\eta, f_w)$ in a Taylor series with respect to $f_w$ about $f_w = f_w^*$,

$$f(\eta, f_w) = f(\eta, f_w^*) + (f_w - f_w^*)\dot{f}(\eta, f_w^*) + \frac{1}{2}(f_w - f_w^*)^2 \ddot{f}(\eta, f_w^*) + \cdots \tag{A.6}$$

If we now let $\eta \to \infty$ and take into account Eq. (A.3c), we obtain

$$f_\infty = f_\infty^* + \frac{1}{2}(f_w - f_w^*)^2 \ddot{f}(\infty, f_w^*) + \cdots \tag{A.7}$$

where $f_\infty^* \equiv f_\infty(\lambda, f_w^*)$. Therefore, the setting of $f''(0) = 0$ is equivalent to $f_\infty$ having a turning point at $f_w = f_w^*$.

The nature of the turning point may now be found by determining $\ddot{f}(\infty, f_w^*)$, which requires a second order analysis, i.e. the solution for $\ddot{f}(\eta)$.

### A.2. Second order analysis

The equation for $\ddot{f}(\eta)$ is,

$$\ddot{f}''' + \frac{1+\lambda}{2}(\ddot{f} f'' + f \ddot{f}'') - 2\lambda f' \ddot{f}' = 2\lambda \dot{f}' \dot{f}' - (1+\lambda)\dot{f} \dot{f}'' \tag{A.8}$$

subject to

$$\ddot{f}(0) = 0, \qquad \ddot{f}'(0) = 0, \qquad \ddot{f}'(\infty) = 0 \tag{A.9a,b,c}$$

The first two boundary conditions (A.9) arise as $f_w$-derivatives of Eqs. (A.3a,b), and Eq. (A.9c) expresses the fact that the entrainment is a constant.

It is possible to find the particular integral and two of the three complementary functions analytically in terms of $f(\eta)$, but we do not need the third. These solutions are

$$\ddot{f}_{\text{pi}} = f'' \quad \Rightarrow \quad \ddot{f}_{\text{pi}}(0) = 0,\ \ddot{f}'_{\text{pi}}(0) = \lambda,\ \ddot{f}_{\text{pi}}(\infty) = 0 \tag{A.10}$$

$$\ddot{f}_{\text{cf1}} = f' \quad \Rightarrow \quad \ddot{f}_{\text{cf1}}(0) = 1,\ \ddot{f}'_{\text{cf1}}(0) = 0,\ \ddot{f}_{\text{cf1}}(\infty) = 0 \tag{A.11}$$

$$\ddot{f}_{\text{cf2}} = \eta f' + f \quad \Rightarrow \quad \ddot{f}_{\text{cf2}}(0) = f_w,\ \ddot{f}'_{\text{cf2}}(0) = 2,\ \ddot{f}_{\text{cf2}}(\infty) = f_\infty^* \tag{A.12}$$

Given the form of Eq. (A.8), the third complementary function must grow linearly as $\eta$ becomes large, and therefore it is inadmissible for the present purpose. The above three components may be added in a suitable manner to obtain a solution which satisfies Eqs. (A.9)

$$\ddot{f} = \ddot{f}_{\text{pi}} - \frac{\lambda}{2}(\ddot{f}_{\text{cf2}} - f_w \ddot{f}_{\text{cf1}}) = f'' - \frac{\lambda}{2}\big[(\eta - f_w) f' + f\big] \tag{A.13}$$

Therefore we may deduce that

$$\ddot{f}(\infty, f_w^*) = -\frac{\lambda}{2} f_\infty^* \tag{A.14}$$

Finally, from Eq. (A.7), we see that the behaviour of $f_\infty$ near to $f_w = f_w^*$ is given by

$$f_\infty = f_\infty^*\left[1 - \frac{\lambda}{4}(f_w - f_w^*)^2 + \cdots\right] \tag{A.15}$$

and therefore this represents a minimum whenever $\lambda < 0$.

## References

[1] P. Cheng, W.J. Minkowycz, Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike, J. Geophys. Res. 82 (1977) 2040–2044.

[2] P. Cheng, The influence of lateral mass flux on free convection boundary layers in a saturated porous medium, Int. J. Heat Mass Transfer 20 (1977) 201–206.

[3] P. Cheng, Heat transfer in geothermal systems, Adv. Heat Transfer 14 (1978) 1–105.

[4] I. Pop, D.B. Ingham, Convective Heat Transfer: Mathematical and Computational Modeling of Viscous Fluids and Porous Media, Pergamon, Oxford, 2001.

[5] D.A. Nield, A. Bejan, Convection in Porous Media, third ed., Springer, New York, 2006.

[6] J.H. Merkin, Free convection boundary layers in a saturated porous medium with lateral mass flux, Int. J. Heat Mass Transfer 21 (1978) 1499–1504.
[7] W.J. Minkowycz, Local non-similar solutions for free convective flow with uniform lateral mass flux in a porous medium, Lett. Heat Mass Transfer 9 (1982) 159–168.
[8] M.A. Chaudhary, J.H. Merkin, I. Pop, Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media. I. Prescribed surface temperature, Europ. J. Mech. B/ Fluids 14 (1995) 217–237.
[9] E. Magyari, B. Keller, Exact analytic solutions for free convection boundary layers on a heated vertical plate with lateral mass flux embedded in a saturated porous medium, Heat Mass Transfer 36 (2000) 109–116.
[10] E. Magyari, I. Pop, B. Keller, New analytical solutions of a well-known boundary value problem in fluid mechanics, Fluid Dynam. Res. 33 (2003) 313–317.
[11] J.H. Merkin, A note on the solution of a differential equation arising in boundary-layer theory, J. Eng. Math. 18 (1984) 31–36.
[12] A. Nakayama, I. Pop, Free convection over a nonisothermal body in a porous medium with viscous dissipation, Int. Comm. Heat Mass Transfer 16 (1989) 173–180.
[13] E. Magyari, D.A.S. Rees, Effect of viscous dissipation of the Darcy free convection boundary-layer flow over a vertical plate with exponential temperature distribution in a porous medium, Fluid Dynam. Res. 38 (2006) 405–429.
[14] E. Magyari, B. Keller, A direct method to calculate the heat transfer coefficient of steady similar boundary layer flows induced by continuous moving surfaces, Int. J. Thermal Sci. 44 (2005) 245–254.
[15] E.M. Sparrow, J.L. Gregg, Similar solutions for free convection from a nonisothermal vertical plate, ASME Trans. 80 (1958) 379–386.
[16] E.R.G. Eckert, R.M. Drake Jr., Analysis of Heat and Mass Transfer, McGraw–Hill, New York, 1972, pp. 306–314.