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Onset of convection in a horizontal porous channel with uniform heat generation using a thermal nonequilibrium model

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Abstract This paper considers the onset of free convection in a horizontal fluid-saturated porous layer with uniform heat generation. Attention is focused on cases where the fluid and solid phases are not in local thermal equilibrium, and where two energy equations describe the evolution of the temperature of each phase. Standard linearized stability theory is used to determine how the criterion for the onset of convection varies with the inter-phase heat transfer coefficient, H, and the porosity-modified thermal conductivity ratio, γ. We also present asymptotic solutions for small values of H. Excellent agreement is obtained between the asymptotic and the numerical results.

Keywords Local thermal non-equilibrium · Instability · Natural convection · Porous media · Internal heat generation

Nomenclature

C	Specific heat
Da	Darcy number
h	Inter-phase heat transfer coefficient
H	Nondimensional inter-phase heat transfer coefficient
g	Gravity
k	Wavenumber of the disturbance
K	Permeability
L	Depth of the convection layer
LTE	Local thermal equilibrium
LTNE	Local thermal nonequilibrium
P	Pressure
q'''	Rate of heat generation

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Q	Overall rate of heat generation
Ra	Darcy–Rayleigh number (Eq. 6)
t	Time
T	Temperature
u, v	Horizontal and vertical velocities
x, y	Horizontal and vertical Cartesian coordinates

Greek symbols

α	Diffusivity ratio
γ	Porosity-scaled conductivity ratio
β	Coefficient of cubical expansion
ρ	Density
σ	Heat capacity ratio
μ	Dynamic viscosity
λ	Constant
ν	Kinematic viscosity
ε	Porosity
ς	Constant denoting internal heating contribution
ψ	Streamfunction
Ψ	Streamfunction disturbance
θ, ϕ	Scaled fluid and solid temperatures
Θ, Φ	Fluid and solid temperature disturbances
ω	Amplification rate of disturbances

Superscripts and subscripts

$\wedge$	Dimensional
basic	Basic state
PM	Porous media
f	Fluid
s	Solid
0	Wall temperature
$\bullet$	k-derivative
$'$	y-derivative

1 Introduction

A recent paper by Nouri-Borujerdi et al. (2006b) has considered the effect of adopting the two-temperature model of thermal conduction in a porous medium on the thermal profiles in a porous layer with internal heat generation. Heat generation may arise in either phase and these authors sought not only to determine the thermal profiles in each phase, but also to determine criteria under which local thermal equilibrium (LTE) may be expected. In these cases the adoption of a single energy equation should then be sufficient to describe the effect of heat generation. When the criteria are not satisfied, then local thermal nonequilibrium (LTNE) occurs, and it is essential to use two separate energy equations. However, the aim of the present paper is to determine the conditions for the marginal stability of these conduction solutions, for the presence of instability will serve to alter the heat transfer characteristics.

There are a significant number of papers which have looked at convection induced by internal heating, but these are dominated by models in which LTE is assumed to be valid. There is an essential difference between flows in rectangular cavities and flows in horizontally unbounded layers. Unless the vertical sidewalls of a cavity are insulated, cavities are susceptible to flow at all nonzero Rayleigh numbers; see Banu et al. (1998) who consider convection in cavities with $T = T_0$ on all four boundaries. Banu et al. (1998) show that this convecting situation is eventually susceptible to instability as Ra increases, the instability taking the form of travelling waves. On the other hand, the basic state for a horizontally unbounded layer is quiescent, and the primary instability is stationary (Gasser and Kazimi 1976). In this situation the onset of convection may be determined by first factoring out a horizontal Fourier component.

Takashima (1989) extended Gasser and Kazimi's analysis to inclined layers. Nouri-Borujerdi et al. (2006b) also extended the same analysis to include the effect of nonzero Darcy number. But much earlier Tveitereid (1977) undertook a detailed stability analysis of fully nonlinear convection above the critical Rayleigh number. The method of Busse (1967), which was used for Bénard convection and subsequently for Darcy–Bénard convection by Strauss (1974), was employed to find the stability of three different steady flows, namely, down-hexagons, up-hexagons and two-dimensional rolls. Up-hexagons were found to be unconditionally unstable, while stable ranges of Rayleigh number for the other two planforms were presented.

However, the present paper is concerned with the effect of adopting separate energy equations for the fluid and solid phases. The only work of which we are aware which has considered such a situation when internal heat generation is present is the paper by Baytas (2003). This paper considered the nonlinear flow in a square cavity which is induced by heat generation in the solid phase. The cavity was subject to a constant temperature on all four sides, and the Forchheimer–Brinkman extended form of Darcy's law was adopted as the momentum equation. Given the boundary conditions, the results of the paper by Baytas (2003) cannot be compared the work we describe here for the reasons given above.

Therefore, we shall follow the analyses of Banu and Rees (2002), Postelnicu and Rees (2003) and Malashetty et al. (2005), who considered the effect of LTNE on Darcy–Bénard convection in an infinite layer, by considering the effect of uniform heat generation. In this regard we shall present numerical solutions for a range of values of H and γ, and supplement this with an asymptotic analysis for small values of H.

2 Governing equations and basic solution

A porous layer of infinite horizontal extent is confined between two parallel rigid surfaces with vertical separation, L, as depicted in Fig. 1. The porous medium is considered to be homogeneous and isotropic. It is assumed that one of the phases generates heat at a uniform rate, and we shall consider each phase in turn. The bounding surfaces are each maintained at the temperature T_0. In this paper, we relax the assumption that the fluid and solid phases of the medium are in local thermal equilibrium and therefore two energy equations are employed, one for each phase. The governing equations of motion for the fluid in the layer are taken to be given by Darcy's law subject to the Boussinesq approximation, and they may be written as,

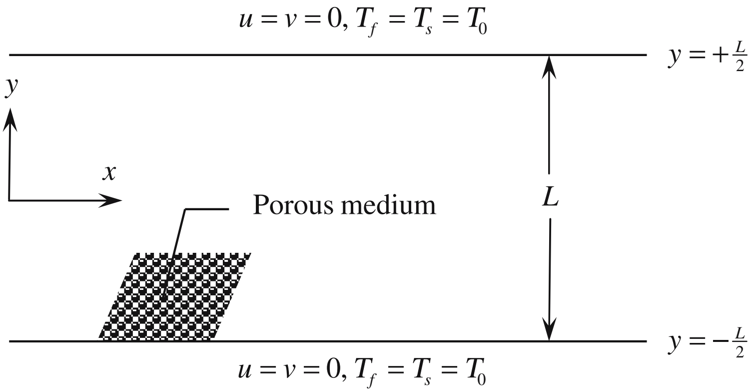

Fig. 1 Definition sketch of the horizontal porous layer with the coordinate system and boundary conditions

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0, \tag{1a}$$

$$\hat{u} = -\frac{K}{\mu}\frac{\partial \hat{P}}{\partial \hat{x}}, \tag{1b}$$

$$\hat{v} = -\frac{K}{\mu}\frac{\partial \hat{P}}{\partial \hat{y}} + \frac{\rho g \beta K}{\mu}(T_f - T_0), \tag{1c}$$

$$(\rho C)_f\left(\varepsilon\frac{\partial T_f}{\partial \hat{t}} + \hat{u}\frac{\partial T_f}{\partial \hat{x}} + \hat{v}\frac{\partial T_f}{\partial \hat{y}}\right) = \varepsilon k_f\left(\frac{\partial^2 T_f}{\partial \hat{x}^2} + \frac{\partial^2 T_f}{\partial \hat{y}^2}\right) + h(T_s - T_f) + \varepsilon q_f''' \cos\varsigma, \tag{1d}$$

$$(1-\varepsilon)(\rho C)_s\frac{\partial T_s}{\partial \hat{t}} = (1-\varepsilon)k_s\left(\frac{\partial^2 T_s}{\partial \hat{x}^2} + \frac{\partial^2 T_s}{\partial \hat{y}^2}\right) + h(T_f - T_s) + (1-\varepsilon)q_s''' \sin\varsigma, \tag{1e}$$

where x and y are the horizontal and vertical coordinates and u and v are the respective velocity components. The value ς allows the contribution of each phase to the overall heat source to be varied mathematically, although we shall consider only $\varsigma = 0$ (heat generation solely in the fluid phase) and $\varsigma = \pi/2$ (heat generation solely in the solid phase). All the other terms have their usual meaning for porous medium convection, and are given in the Nomenclature. We note that there is no need to include the Forchheimer terms in the above formulation since their presence has no effect on the linear stability analysis presented here. In similar fashion we also omit the Brinkman terms. Most porous media are characterized by very small values of the Darcy number, and since the results of Rees (2002) show that the critical Darcy–Rayleigh number is altered only by an $O(Da^{1/2})$ amount, it is clear that the adoption of the Darcy model without the Brinkman terms provides accurate stability criteria in most cases. Finally, we point out that the inter-phase heat transfer coefficient, h, is a function of the detailed geometry of the porous medium and of the conductivities and diffusivities of the fluid and solid phases. As our analysis is a linear stability theory the present values of h correspond to zero microscopic Reynolds numbers. Here we shall consider the effect of different values of h which are assumed to be known.

The basic state corresponding to x-independent steady solutions of Eq. 1d, e have been presented in detail in a recent paper by Nouri-Borujerdi et al. (2006b). It is the stability of this basic state that forms the subject of the present paper. Eq. 1a–d may be nondimensionalized using the following substitutions,

$$\hat{t} = \frac{L^2\sigma}{\alpha_{\mathrm{PM}}}t, \quad (\hat{x},\hat{y}) = L(x,y), \qquad (\hat{u},\hat{v}) = \frac{\alpha_{\mathrm{PM}}}{L}(u,v) \tag{2a}$$

and

$$\hat{P} = \frac{\alpha_{\mathrm{PM}}\mu}{K}P, \quad (T_f,T_s) = T_0 + \frac{QL^2}{k_{\mathrm{PM}}}(\theta,\phi), \tag{2b}$$

where

$$\sigma = \frac{(\rho C)_{\mathrm{PM}}}{(\rho C)_f}, \alpha_{\mathrm{PM}} = \frac{k_{PM}}{(\rho C)_f}, k_{\mathrm{PM}} = \varepsilon k_f + (1-\varepsilon)k_s,$$
$$Q^2 = [\varepsilon\, q_f''']^2 + [(1-\varepsilon)q_s'''\,]^2. \tag{2c}$$

These transformations yield the following system of equations,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3a}$$

$$u = -\frac{\partial P}{\partial x}, \tag{3b}$$

$$v = -\frac{\partial P}{\partial y} + Ra\theta, \tag{3c}$$

$$\left(\frac{\gamma+1}{\gamma+\alpha}\right)\frac{\partial\theta}{\partial t} + \left(\frac{\gamma+1}{\gamma}\right)\left(u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right) = \left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + H\,(\phi-\theta) + \left(\frac{\gamma+1}{\gamma}\right)\cos\,\varsigma, \tag{3d}$$

$$\alpha\left(\frac{\gamma+1}{\gamma+\alpha}\right)\frac{\partial\phi}{\partial t} = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + H\,(\theta-\phi) + (\gamma+1)\sin\,\varsigma. \tag{3e}$$

In above equations the non-dimensional parameters, H, α, γ and Ra are the non-dimensional inter-phase heat transfer parameter, the diffusivity ratio, the porosity-modified conductivity ratio, and the Rayleigh number, respectively; they are defined as follows,

$$H = \frac{hL^2}{\varepsilon k_f}, \quad \alpha = \frac{\alpha_f}{\alpha_s}, \quad \gamma = \frac{\varepsilon k_f}{(1-\varepsilon)k_s} \quad \text{and} \quad Ra = \frac{g\beta KQL^3}{\nu\alpha_{\mathrm{PM}}k_{\mathrm{PM}}}. \tag{4}$$

The appropriate boundary conditions are that,

$$v(x,\pm\tfrac{1}{2}) = \theta(x,\pm\tfrac{1}{2}) = \phi(x,\pm\tfrac{1}{2}) = 0. \tag{5}$$

From the continuity Eq. 3a, the streamfunction ψ may be defined according to,

$$u = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad v = \frac{\partial\psi}{\partial x}. \tag{6}$$

After eliminating the pressure P between Eq. 3b, c, Eq. 3a–e reduce to the system,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=Ra\frac{\partial\theta}{\partial x}, \tag{7a}$$

$$\left(\frac{\gamma+1}{\gamma+\alpha}\right)\frac{\partial\theta}{\partial t}+\left(\frac{\gamma+1}{\gamma}\right)\left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}\right)$$
$$=\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)+H(\phi-\theta)+\left(\frac{\gamma+1}{\gamma}\right)\cos\varsigma \tag{7b}$$

$$\alpha\left(\frac{\gamma+1}{\gamma+\alpha}\right)\frac{\partial\phi}{\partial t}=\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+H(\theta-\phi)+(\gamma+1)\sin\varsigma, \tag{7c}$$

which are to be solved subject to the boundary conditions,

$$\psi(x,\pm\tfrac{1}{2})=\theta(x,\pm\tfrac{1}{2})=\phi(x,\pm\tfrac{1}{2})=0. \tag{8}$$

In the absence of convection the basic steady state is such that heat which is generated within the layer is removed by conduction through the upper and lower surfaces. When the heat generation is in the fluid phase ($\varsigma=0$) the basic temperature distributions are

$$\theta_{\text{basic}}=\frac{\gamma\lambda^2+8}{8\gamma\lambda^2}-\frac{\cosh(\lambda y)}{\gamma\lambda^2\cosh(\lambda/2)}-\frac{y^2}{2} \tag{9a}$$

$$\phi_{\text{basic}}=\frac{\lambda^2-8}{8\lambda^2}+\frac{\cosh(\lambda y)}{\lambda^2\cosh(\lambda/2)}-\frac{y^2}{2} \tag{9b}$$

and, when the heat generation is in the solid phase ($\varsigma=\pi/2$), they are

$$\theta_{\text{basic}}=\frac{\lambda^2-8}{8\lambda^2}+\frac{\cosh(\lambda y)}{\lambda^2\cosh(\lambda/2)}-\frac{y^2}{2} \tag{10a}$$

$$\phi_{\text{basic}}=\frac{\lambda^2+\gamma}{8\lambda^2}-\frac{\gamma\cosh(\lambda y)}{\lambda^2\cosh(\lambda/2)}-\frac{y^2}{2}, \tag{10b}$$

where

$$\lambda^2=H(\gamma+1) \tag{11}$$

has been defined for convenience. It is the stability of these basic states that forms the subject of the present paper.

3 Linear stability theory

We may assess the stability characteristics of the evolving basic state using a straightforward perturbation theory. Therefore we set,

$$\psi=\Psi(y)e^{\omega t}\cos kx, \tag{12a}$$

$$\theta=\theta_{\text{basic}}(y)+\Theta(y)e^{\omega t}\sin kx, \tag{12b}$$

$$\phi=\phi_{\text{basic}}(y)+\Phi(y)e^{\omega t}\sin kx, \tag{12c}$$

where Ψ, Θ and Φ are of sufficiently small amplitude that nonlinear terms may be neglected. The value, k, is the horizontal wavenumber of the disturbances. The principle of exchange of stabilities may be shown easily, since the resulting equations may be put into a self-adjoint form, thereby indicating that the onset of convection is stationary. Therefore, we may setω=0. The linearized stability equations are now,

$$\Psi'' - k^2\Psi = Rak\Theta, \tag{13a}$$

$$\Theta'' - k^2\Theta + H(\Phi - \Theta) = -\left(\frac{\gamma+1}{\gamma}\right) k\Psi\theta'_{\text{basic}}, \tag{13b}$$

$$\Phi'' - k^2\Phi + H\gamma(\Theta - \Phi) = 0, \tag{13c}$$

which are subject to the boundary conditions,

$$\Psi(\pm\tfrac{1}{2}) = \Theta(\pm\tfrac{1}{2}) = \Phi(\pm\tfrac{1}{2}) = 0 \tag{14}$$

In Eq. 13a–c primes denote differentiation with respect to y. As these equations cannot be solved analytically it is necessary to resort to numerical methods.

4 Numerical method

Equation 13a–c forms an ordinary differential eigenvalue problem for Ra as a function of the inter-phase heat transfer coefficient, H, the porosity modified conductivity ratio, γ, and the wavenumber, k. When $H \to \infty$, we recover the local thermal equilibrium case first studied by Gasser and Kazimi (1976). In this paper, we use a direct method of numerical solution of similar form to that used by Rees (2002).

Equation 13a–c were discretised using fourth order compact differences on a uniform grid in the y-direction with 200 intervals. The zero streamfunction and temperature conditions provide a sufficient number of boundary conditions for these equations. However, the eigenvalue, Ra, also needs to be found, and this requires one more condition, the normalization condition,

$$\Theta'(\tfrac{1}{2}) = -1, \tag{15}$$

which simply fixes the amplitude of the eigensolution. Apart from the fact that fourth order compact differences have been used, the multi-dimensional Newton–Raphson iteration matrix which is obtained has exactly the same form as that given in Rees (2002). Specifically it has block tridiagonal form with an extra row and column of blocks. These latter arise from the presence of the eigenvalue and the normalization condition. A suitably modified block tridiagonal matrix algorithm was used to solve this discretised system.

We found that the neutral stability curves relating the critical value of Ra to k always have a single minimum, and therefore we concentrate solely on the computation of this minimum. This is achieved numerically by insisting that $\partial Ra/\partial k = 0$ and by supplementing Eq. 15a–c by their derivatives with respect to k. Thus, if we define,

$$\overset{\bullet}{\Psi} = \frac{\partial\Psi}{\partial k}, \quad \overset{\bullet}{\Theta} = \frac{\partial\Theta}{\partial k} \quad \text{and} \quad \overset{\bullet}{\Phi} = \frac{\partial\Phi}{\partial k}, \tag{16}$$

then the differentiation of Eq. 13a–c with respect to k yields the following system,

$$\overset{\bullet}{\Psi}'' - k^2 \overset{\bullet}{\Psi} - Rak \overset{\bullet}{\Theta} = 2k\Psi + Ra\Theta, \tag{17a}$$

$$\overset{\bullet}{\Theta}'' - k^2 \overset{\bullet}{\Theta} + H(\overset{\bullet}{\Phi} - \overset{\bullet}{\Theta}) + k\left(\frac{\gamma+1}{\gamma}\right)\theta'_{\text{basic}} \overset{\bullet}{\Psi} = 2k\Theta - \left(\frac{\gamma+1}{\gamma}\right)\theta'_{\text{basic}}\Psi, \tag{17b}$$

$$\overset{\bullet}{\Phi}'' - k^2 \overset{\bullet}{\Phi} + \gamma H(\overset{\bullet}{\Theta} - \overset{\bullet}{\Phi}) = 2k\Phi, \tag{17c}$$

which are subject to the boundary conditions,

$$\overset{\bullet}{\Psi}(\pm\tfrac{1}{2}) = \overset{\bullet}{\Theta}(\pm\tfrac{1}{2}) = \overset{\bullet}{\Phi}(\pm\tfrac{1}{2}) = 0. \tag{18}$$

The wavenumber is now a second eigenvalue and therefore it is necessary to impose a second normalization condition. We have chosen to use $\overset{\bullet}{\Theta}'(\frac{1}{2}) = -1$, but any other value of this derivative would yield precisely the same values of Ra and k. There are now only two parameters to vary and solutions are presented for different values of γ and H and for both the chosen values of ς.

5 Results

Figures 2 and 3 show some typical disturbance streamlines and isotherms ($\Psi \cos kx$, $\Theta \sin kx$ and $\Phi \sin kx$) for the problem being solved. Both figures correspond to $H = 100$ and $\gamma = 0.01$ but the former has internal heating in the fluid phase while the latter has it in the solid phase. In both cases, the disturbances tend to lie in the top half of the layer since the layer is unstably stratified only in that half.

The respective critical values are $Ra = 1.3151, k = 9.9518$ and $Ra = 12.321, k = 8.058$. There is little difference in the critical wavenumbers because the shape of the basic conducting state does not vary greatly. But there is a substantial difference in the critical Rayleigh number. The reason for this is that the amplitude of the basic temperature field when the solid is the heat generating phase is much less than when the fluid is the heat generating phase. This means that the Rayleigh number must be much larger in the solid-heating case in order to ensure that buoyancy forces are sufficiently strong. These figures also show that the disturbance temperature fields corresponding to the two phases have different thicknesses. That corresponding to the solid phase is thicker than the one for the fluid phase, but this is true only when $\gamma < 1$. When H takes larger values, then LTE is approached and the disturbance fields become identical in both profile and magnitude.

In Figures 4 and 5 we show the variation of critical Rayleigh number and wavenumber with the inter-phase heat transfer coefficient, H, for specific values of γ when heat generation takes place in the fluid phase. It is clear from these figures that both Ra and k approach a constant which is independent of γ as $H \to \infty$. This is, of course, the LTE limit, and the numerical results match well with the equivalent critical values obtained from the LTE equations: $Ra = 471.3787$ and $k = 4.67519$. At the opposite extreme, namely as $H \to 0$, the Ra curves tend to a constant which depends strongly on the value of γ, but the asymptotic value of k is again independent of γ. These behaviours are studied in detail in the Appendix where it is shown that the limiting

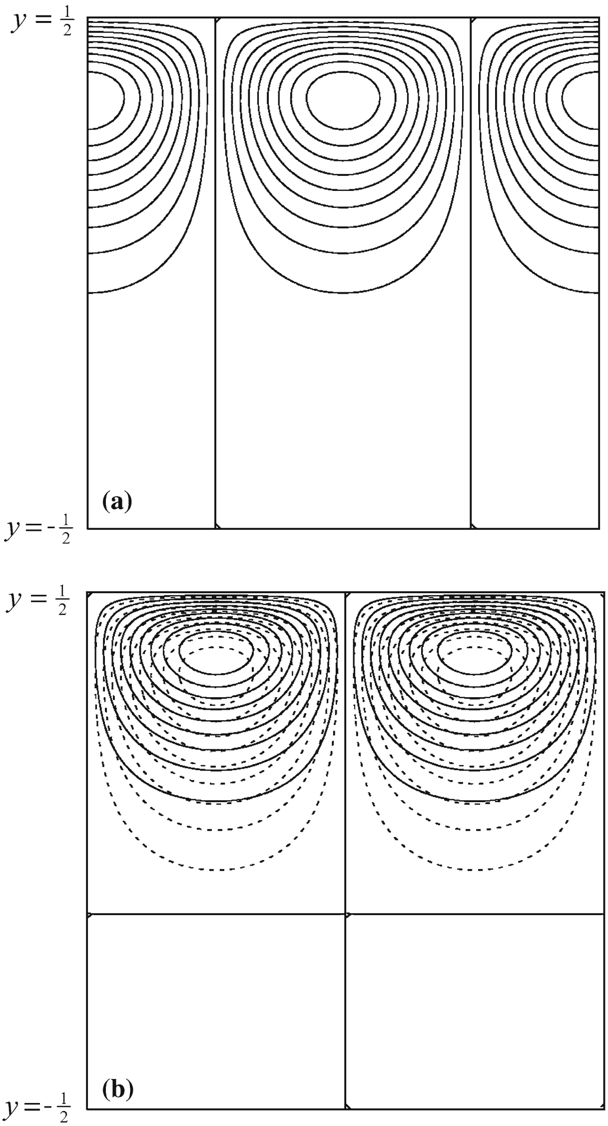

Fig. 2 Disturbance streamlines and isotherms corresponding to $H = 100$ and $\gamma = 0.01$ with internal heating in the fluid phase ($\varsigma = 0$): (**a**) streamlines, (**b**) isotherms. The *solid* and *dashed lines* represent fluid and solid temperature disturbances, respectively

value of k is exactly the same as for LTE, while that for Ra is

$$Ra = \frac{471.3787\gamma^2}{(\gamma+1)^2} + O(H). \tag{19}$$

At intermediate values of H the critical value of Ra increases with increasing H. However, the critical value of k first increases and then decreases with H, and the maximum value that k achieves becomes quite large as γ decreases. These features

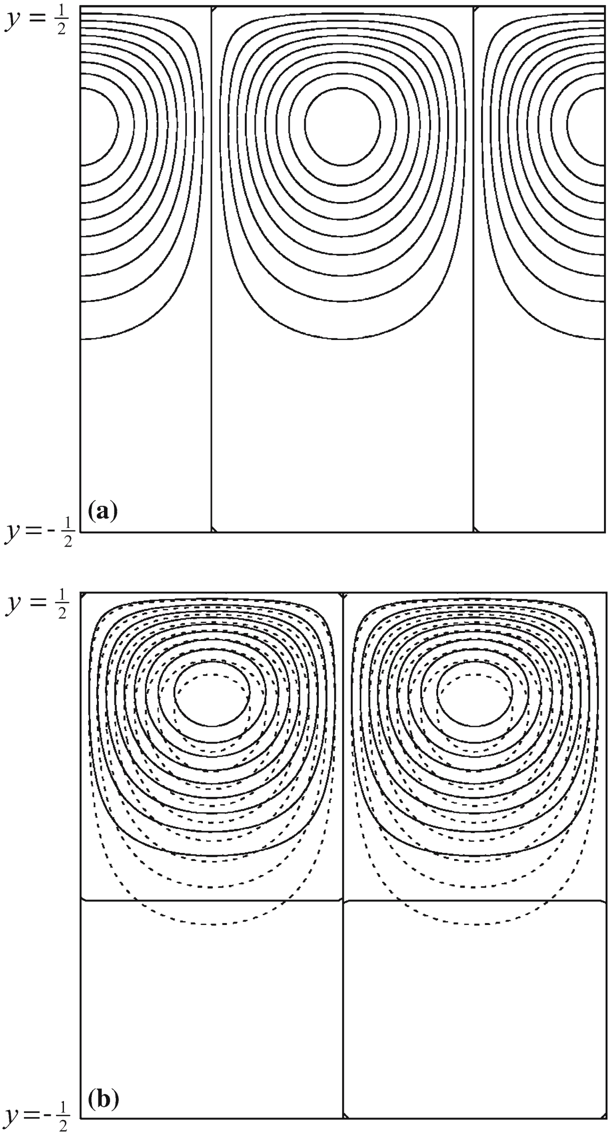

Fig. 3 Disturbance streamlines and isotherms corresponding to $H = 100$ and $\gamma = 0.01$ with internal heating in the solid phase ($\varsigma = \pi/2$): (**a**) streamlines, (**b**) isotherms. The *solid* and *dashed lines* represent fluid and solid temperature disturbances, respectively

appear to be common in this type of stability problem, see Banu and Rees (2002), Postelnicu and Rees (2003) and Malashetty et al. (2005). We also note that large values of γ also cause LTE to be approached for any chosen value of H.

Figures 6 and 7 display the equivalent behaviours of the critical values of Ra and k when heating takes place in the solid phase. As above, LTE is achieved as H is increased for constant values of γ, and also as γ increases for constant values of H. However, as $H \to 0$, the critical values of Ra rise rapidly, and Fig. 6 seems to indicate

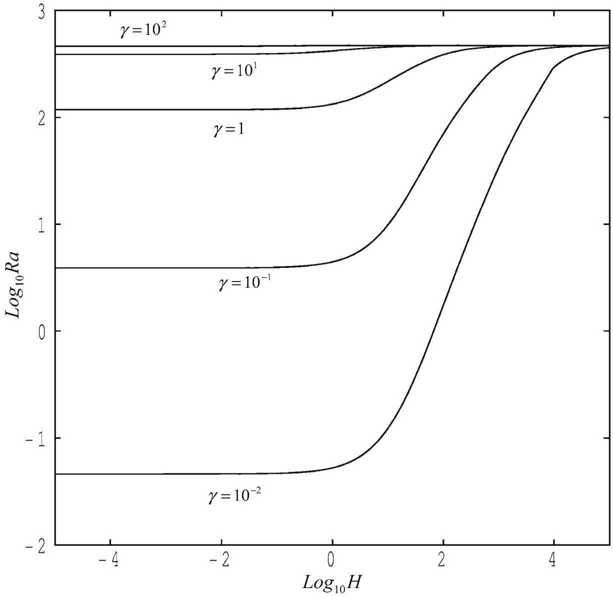

Fig. 4 Variation of the critical Rayleigh number with $Log_{10}\,H$ for various values of γ and with internal heating in the fluid phase ($\varsigma = 0$)

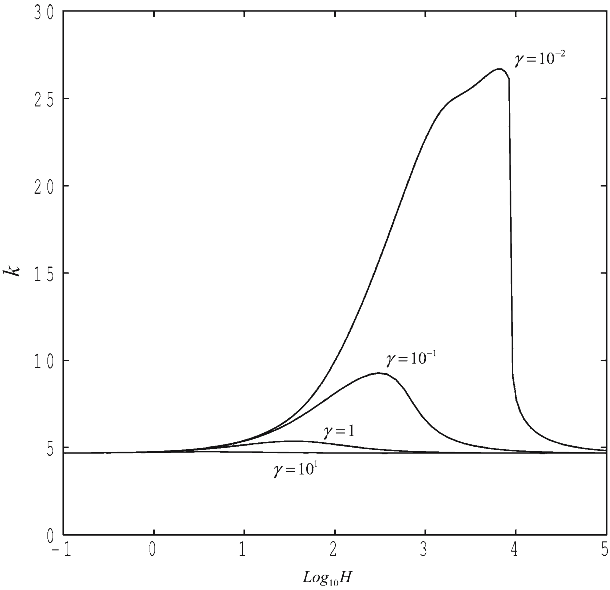

Fig. 5 Variation of the critical wavenumber, k, with $Log_{10}\,H$ for various values of γ and with internal heating in the fluid phase ($\varsigma = 0$)

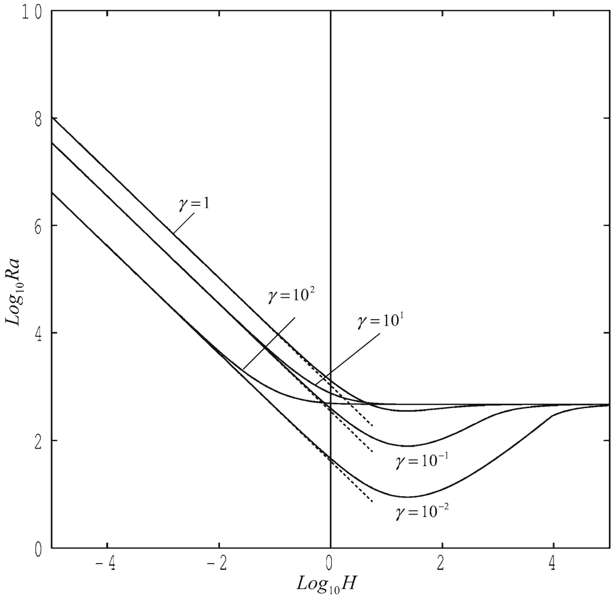

Fig. 6 Variation of the critical Rayleigh number with $Log_{10}\, H$ for various values of γ with internal heating in the solid phase ($\varsigma = \pi/2$). *Dashed lines* indicate asymptotic solutions given by Eq. A11

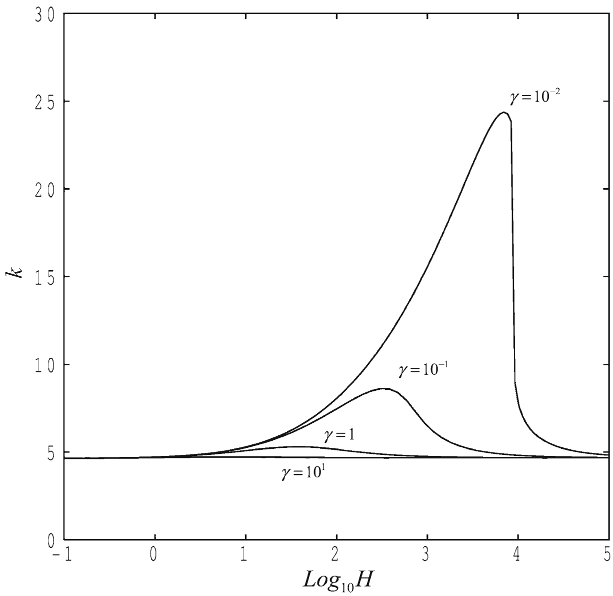

Fig. 7 Variation of critical wavenumber, k, for various values of γ and with internal heating in the solid phase ($\varsigma = \pi/2$)

that Ra is inversely proportional to H. More intriguingly, the asymptotic curves for $\gamma = c$ and $\gamma = c^{-1}$ appear to be identical for any chosen value of the constant c. Again, an asymptotic analysis of this stability problem for small values of H may be found in the Appendix where it is found that

$$Ra \sim \frac{4214.867}{H\left(\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}\right)^2}. \tag{20}$$

at leading order. This expression confirms not only the dependence on H^{-1}, but also the unusual γ-dependence. This asymptotic reslation is also included in Fig. 6 for reference. The other feature of the asymptotic analysis is that k ~4.6930 at leading order, which is slightly different from the value corresponding to LTE. At intermediate values of H it is found that the variation in the critical value of Ra is not monotonic for small values of γ.

6 Conclusion

In this paper, we have considered the linear instability of the conductive state in a horizontal porous layer which is caused by uniform internal heat generation in either the fluid or the solid phase. Unimodal neutral stability curves arise for this configuration and we have presented numerical solutions for the critical Rayleigh number as a function of both H and γ, together with the values of the wavenumber, k, which minimize the Rayleigh number. Not surprisingly, we find that LTE is achieved whenever either H or γ is sufficiently large, as such conditions serve to increase the dominance of the source/sink terms in the two energy equations. However, when the heat is being generated in the fluid phase, the equation for the disturbance temperature in the solid phase decouples from the other disturbance equations leaving a scaled version of the LTE problem to solve. On the other hand, when the heat generation is in the solid phase, a small value of H means that the fluid phase is hardly affected by the internal heating. In turn, this means that the critical Rayleigh number must be correspondingly large in order to obtain instability.

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Appendix: asymptotic analysis for small values of *H*

Case 1 When heat generation is within the fluid phase, $\varsigma = 0$.

According to Fig. 4 the critical Rayleigh number is constant when H is small, but with a value that depends on γ. If we expand θ_{basic} given in (9a) as a power series in small values of H, then we obtain,

$$\theta_{\text{basic}} = \left(\frac{\gamma+1}{\gamma}\right)\left(\frac{1}{4} - y^2\right) + O(H). \tag{A1}$$

The leading order form of the linearized disturbance equations (15a–c) are given by the following system,

$$\Psi'' - k^2\Psi = Rak\Theta, \tag{A2a}$$

$$\Theta'' - k^2\Theta - \left(\frac{\gamma+1}{\gamma}\right)^2 k\Psi y = 0, \tag{A2b}$$

$$\Phi'' - k^2\Phi = 0. \tag{A2c}$$

It is clear that (A2c) is decoupled from (A2a) and (A2b), and has solution $\Phi = 0$. We may now reduce the remaining equations to the standard LTE form by first defining,

$$\hat{\Psi} = \left(\frac{\gamma+1}{\gamma}\right)^2 \Psi. \tag{A3}$$

Equation (A2b, c) now reduce to the system

$$\hat{\Psi}'' - k^2\hat{\Psi} = \left(\frac{\gamma+1}{\gamma}\right)^2 Rak\Theta, \tag{A4a}$$

$$\Theta'' - k^2\Theta - k\hat{\Psi}y = 0, \tag{A4b}$$

where the coefficient of $k\Theta$ plays the same role as the Rayleigh number does in the LTE analysis of Nouri-Borujerdi et al. (2006b), i.e. where the linearised stability equations are,

$$\Psi'' - k^2\Psi = Ra_{\mathrm{LTE}}k\Theta, \tag{A5a}$$

$$\Theta'' - k^2\Theta - k\Psi y = 0. \tag{A5b}$$

According to Nouri-Borujerdi et al. (2006b) the critical Rayleigh number, Ra_{LTE} for the case of LTE is 471.3787. Therefore, it is clear that the critical Rayleigh number for Eq. A4 must be,

$$Ra = \frac{471.3787\gamma^2}{(\gamma+1)^2} + O(H). \tag{A6}$$

The corresponding critical wavenumber is $k = 4.67519$, which is in excellent agreement with our computed data for small H.

Case 2 When heat generation is in the solid phase, $\varsigma = \pi/2$.

Figure 6 shows that the critical value of Ra increases as H decreases. Numerically, the slope of these curves is -1, and this suggests that RaH tends to a γ-dependent constant as $H \to 0$. In the small H limit, the magnitude of the basic thermal profile of the fluid phase is of O(H), and is,

$$\theta_{\mathrm{basic}} = H(\gamma+1)\left(\frac{y^3}{6} - \frac{y}{8}\right) + O(H^2). \tag{A7}$$

In order to write down a consistent set of linearized stability equations in the small-H limit, it is necessary first to rescale Ψ and Ra as follows,

$$\overline{Ra} = \frac{HRa(\gamma+1)^2}{\gamma} \quad \text{and} \quad \hat{\Psi} = \frac{H(\gamma+1)^2}{\gamma}\Psi. \tag{A8}$$

At leading order the linearized stability equations are now,

$$\hat{\Psi}'' - k^2\hat{\Psi} = \overline{Ra}k\Theta, \tag{A9a}$$

$$\Theta'' - k^2\Theta + k\hat{\Psi}\left(\frac{y^3}{6} - \frac{y}{8}\right) = 0, \tag{A9b}$$

$$\Phi'' - k^2\Phi = 0. \tag{A9c}$$

It is clear that the above equations are independent of both H and γ. Again $\Phi = 0$, but now the resulting system possesses the minimizing solution,

$$\overline{Ra} = 4214.867 \quad \text{and} \quad k = 4.6390. \tag{A10}$$

Hence the critical Rayleigh number in the small-H limit is

$$Ra \sim \frac{4214.867\gamma}{H(\gamma+1)^2} = \frac{4214.867}{H\left(\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}\right)^2}. \tag{A11}$$

Asymptotic curves corresponding to this formula are shown in Fig. 6 as dashed lines. Given that (A11) gives exactly the same values for Ra when γ=constant and when γ^{-1} = constant, this explains why we get the unusual merging of the curves for $\gamma = 10^1$ and $\gamma = 10^{-1}$, for example.

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