ORIGINAL PAPER

The effect of local thermal non-equilibrium on conduction in porous channels with a uniform heat source

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Abstract   We examine the effect of local thermal non-equilibrium on the steady state heat conduction in a porous layer in the presence of internal heat generation. A uniform source of heat is present in either the fluid or the solid phase. A two-temperature model is assumed and analytical solutions are presented for the resulting steady-state temperature profiles in a uniform porous slab. Attention is then focussed on deriving simple conditions which guarantee local thermal equilibrium.

Nomenclature

c	Specific heat
C_1, C_2, C_3	Constants
h	Inter-phase heat transfer coefficient
H	Non-dimensional inter-phase heat transfer coefficient
k	Thermal conductivity
L	Length scale
q'''	Rate of heat generation per unit volume
Q	Overall rate of heat generation
t	Time
T_f, T_s	Fluid and solid temperatures, respectively
T_0	Ambient temperature
$\underline{\upsilon}$	Velocity vector

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x	Coordinate across the channel
y	Coordinate along the channel

Greek

γ	Porosity-scaled conductivity ratio
δ	Constant defined by Eq. 22
ϵ	Porosity
ζ	Constant denoting internal heating contributions
η	Similarity variable
θ	Non-dimensional fluid temperature
λ	Constant
ρ	Density
τ	Scaled time
ϕ	Non-dimensional solid temperature
ω	Non-dimensional diffusivity

Subscripts and superscripts

f	Fluid
s	Solid
ˆ	Dimensional

1 Introduction

Much effort has been devoted recently to determining conditions which guarantee that the assumption of Local Thermal Equilibrium (LTE) is accurate when modelling heat transfer in porous media. When it is accurate, then the thermal field is well-approximated by a single thermal energy equation. In other circumstances Local Thermal Non-equilibrium (LTNE) prevails, and it is necessary to employ two energy equations, one for each phase.

Nield and Bejan (2006) have presented the following equations for the two-temperature model,

$$\epsilon(\rho c)_{\mathrm{f}}\frac{\partial T_{\mathrm{f}}}{\partial t} + (\rho c)_{\mathrm{f}}\underline{\upsilon}.\nabla T_{\mathrm{f}} = \epsilon\nabla.(k_{\mathrm{f}}\nabla T_{\mathrm{f}}) + h(T_{\mathrm{s}} - T_{\mathrm{f}}) + \epsilon q_{\mathrm{f}}''', \tag{1}$$

$$(1-\epsilon)(\rho c)_{\mathrm{s}}\frac{\partial T_{\mathrm{s}}}{\partial t} = (1-\epsilon)\nabla.(k_{\mathrm{s}}\nabla T_{\mathrm{s}}) + h(T_{\mathrm{f}} - T_{\mathrm{s}}) + (1-\epsilon)q_{\mathrm{s}}'''. \tag{2}$$

In these equations ϵ is the porosity of the porous medium, q denotes the rate of heat gain per unit volume, and the subscripts, f and s, denote fluid and solid properties, respectively. The variable h is an inter-phase heat transfer coefficient. In Eqs. 1 and 2 we follow the standard practice that T_{f} and T_{s} are intrinsic averages of the temperature fields, and this allows us to set $T_{\mathrm{f}} = T_{\mathrm{s}} = T_0$ whenever the boundary of the porous medium is maintained at the temperature T_0. On the other hand, $\underline{\upsilon}$ is a superficial average.

Nield (1998) considered steady conduction in a porous medium subject to an imposed temperature on the boundary of the domain, and where the conductivities of the phases are uniform. He showed that $T_{\mathrm{f}} = T_{\mathrm{s}}$ by deriving a Helmholtz equation for $T_{\mathrm{f}} - T_{\mathrm{s}}$ which is subject to $T_{\mathrm{f}} - T_{\mathrm{s}} = 0$ on the boundary. Therefore, LTE always occurs in steady conduction problems where the boundary temperature, rather than a heat flux, is imposed.

Very recently the present authors have considered unsteady conduction where the plane boundary of a semi-infinite porous medium has its temperature raised suddenly (A. Nouri-Borujerdi et al., submitted). If one were to assume LTE, then the resulting temperature field is described by the complementary error function of a similarity variable. When LTNE is assumed and when the diffusivities of the phases are different from one another, the thermal fields in the two phases at early times correspond to complementary error functions with different arguments, and therefore strong LNTE effects are observed. However, as time progresses, the thermal front slows down, as its speed is inversely proportional to $t^{1/2}$, and LTE is eventually established. In this case rapid changes in boundary conditions result in LTNE in an unsteady situation.

Al-Nimr and Abu-Hijleh (2002) studied a channel flow problem using the equations of Schumann (1929), i.e. Eqs. 1 and 2 without diffusion. At $t = 0$ the inlet temperature was raised suddenly and the authors investigated the subsequent change in the temperature field. At long times the problem becomes like that of Nield (1998) which is described at the beginning of this section, and therefore LTE is always achieved eventually. Therefore, attention is focussed on the thermal relaxation time over which LTNE gives way to LTE. The final conclusions are lengthy, but it is found that the relaxation time is inversely proportional to a volumetric Biot number.

Similar conclusions with regard to the rapidity of changing conditions have been made by Minkowycz et al. (1999) and Al-Nimr and Kiwan (2002). However, it remains possible for strong LTNE effects to occur in steady flows. This was shown conclusively in the papers by Rees and Pop (2000) and Rees (2003). Both papers studied free convection boundary layer flow from a uniformly hot vertical surface, but the former adopted a boundary layer approach while the latter solved the full elliptic system of governing equations. In these papers the distance from the leading edge appears to play the same role as time in the above-quoted paper by A. Nouri-Borujerdi et al. (submitted) in that LTE is recovered as x increases. However, near the leading edge, there is strong inflow from the cold external region which decreases the thickness of the thermal boundary layer in the fluid phase compared with that of the solid phase, in which conduction but not advection takes place.

In the present paper, we describe a steady state conduction problem in a stagnant porous medium which can display strong LTNE effects. Although this appears to contradict the conclusion of Nield (1998) quoted above, it does not do so in fact; below we will show that a consequence of the presence of internal heat generation is to render inhomogeneous Nield's Helmholtz equation, and this forces LTNE.

2 Governing equations

We consider a fluid-saturated porous channel of width L in the x-direction whose boundaries are held at the temperature $T = T_0$. One of the phases generates heat at a uniform rate, and we shall consider each in turn. When the channel is sufficiently long conduction takes place solely in the $\hat{x}$-direction and therefore we may suppress $\hat{y}$-derivatives in (1) and (2). In general flow will take place in the $\hat{y}$-direction, but this does not affect the steady-state temperature field. Therefore, the following equations governing the steady state temperature profiles,

$$\epsilon k_{\mathrm{f}} \frac{\mathrm{d}^2 T_{\mathrm{f}}}{\mathrm{d}\hat{x}^2} + h(T_{\mathrm{s}} - T_{\mathrm{f}}) + \epsilon q_{\mathrm{f}}''' = 0, \tag{3}$$

$$(1 - \epsilon) k_s \frac{\mathrm{d}^2 T_{\mathrm{s}}}{\mathrm{d}\hat{x}^2} + h(T_{\mathrm{f}} - T_{\mathrm{s}}) + (1 - \epsilon) q_{\mathrm{s}}''' = 0, \tag{4}$$

and these are to be solved subject to

$$T_{\mathrm{f}} = T_{\mathrm{s}} = T_0 \quad \text{at} \quad \hat{x} = \pm L/2. \tag{5}$$

We non-dimensionalize Eqs. 3 and 4 using the following substitutions,

$$\hat{x} = Lx, \quad (T_{\mathrm{f}}, T_{\mathrm{s}}) = T_0 + \frac{QL^2}{\epsilon k_{\mathrm{f}}} (\theta, \phi), \tag{6}$$

where Q is given by

$$Q^2 = [\epsilon q_{\mathrm{f}}''']^2 + [(1 - \epsilon) q_{\mathrm{s}}''']^2, \tag{7}$$

and where we may define a phase angle, ζ, according to

$$\epsilon q_{\mathrm{f}}''' = Q \cos \zeta, \quad (1 - \epsilon) q_{\mathrm{s}}''' = Q \sin \zeta. \tag{8}$$

The value ζ allows the contribution of each phase to the overall heat source to be varied mathematically; although we shall consider only $\zeta = 0$ and $\zeta = \pi/2$ (i.e. heat generation in only one of the phases) since solutions for other values of ζ may be obtained easily by superposition because (3) and (4) are linear. We obtain the following equations,

$$\theta'' + H(\phi - \theta) + \cos \zeta = 0, \tag{9}$$

$$\phi'' + H\gamma(\theta - \phi) + \gamma \sin \zeta = 0, \tag{10}$$

where primes denote derivatives with respect to x since the resulting solution will be independent of y but close to the ends of a long channel. The two non-dimensional parameters, H and γ are the non-dimensional inter-phase heat transfer parameter and the porosity modified conductivity ratio:

$$H = \frac{hL^2}{\epsilon k_{\mathrm{f}}}, \quad \gamma = \frac{\epsilon k_{\mathrm{f}}}{(1 - \epsilon) k_{\mathrm{s}}}. \tag{11}$$

Equations 9 and 10 are to be solved subject to

$$\theta = \phi = 0 \quad \text{on} \quad x = \pm \frac{1}{2}. \tag{12}$$

It is useful to follow Nield's (1998) analysis and derive the equation which is satisfied by the difference between the temperatures of the phases. Equations 9 and 10 may be manipulated to yield

$$(\theta - \phi)'' - H(1 + \gamma)(\theta - \phi) = \gamma \sin \zeta - \cos \zeta, \tag{13}$$

which may be solved subject to the boundary conditions,

$$\theta - \phi = 0 \quad \text{on} \quad x = \pm \frac{1}{2}. \tag{14}$$

We shall not solve this equation, but it is clear that the solution is nonzero, and therefore LTNE occurs even in the steady state.

3 Analytical solutions and discussion

Equations 9 and 10 may be solved analytically. For general values of ζ the solution may be written in the form,

$$\theta = C_1 - C_2 x^2 + C_3 \cosh(\lambda x), \tag{15}$$

$$\phi = C_1 + \frac{(2C_2 - \cos \zeta)}{H} - C_2 x^2 - \gamma C_3 \cosh(\lambda x), \tag{16}$$

where the constants, C_1, C_2, C_3 and λ are given by

$$C_1 = \frac{H\gamma(\gamma + 1)(\sin \zeta + \cos \zeta) + 8(\cos \zeta - \gamma \sin \zeta)}{8H(\gamma + 1)^2}, \tag{17}$$

$$C_2 = \frac{\gamma(\sin \zeta + \cos \zeta)}{2(\gamma + 1)}, \tag{18}$$

$$C_3 = \frac{\gamma \sin \zeta - \cos \zeta}{H(\gamma + 1)^2 \cosh(\lambda/2)}, \tag{19}$$

and

$$\lambda = \sqrt{H(\gamma + 1)}. \tag{20}$$

Detailed solution curves are not presented since they are roughly parabolic in shape for all values of H and γ. When λ is small the cosh function is roughly parabolic, but when λ is large (and hence $H(\gamma + 1)$ is large) the cosh term is of $O(H^{-1})$ relative to the constant and parabolic terms, with a narrow boundary layer near to $x = \pm\frac{1}{2}$.

We are now in a position to determine how closely LTE is approximated over a wide range of values of H and γ. The solutions are symmetric about $x = 0$, the centre of the channel, and this is where the maximum difference between the temperatures of the phases occurs. We define δ to be proportional to the maximum relative difference between the temperatures:

$$\delta = \left| \frac{\theta - \phi}{\theta + \phi} \right|_{x=0}. \tag{21}$$

When the heat generation is in the fluid phase, for which $\zeta = 0$, we obtain

$$\delta = \frac{4\lambda^2[\cosh(\lambda/2) - 1]}{[H(\lambda^2\gamma + 8) - 4\lambda^2]\cosh(\lambda/2) + 4H(\gamma - 1)}, \tag{22}$$

and when it is in the solid phase, for which $\zeta = \pi/2$, we have

$$\delta = \frac{4(\gamma + 1)[\cosh(\lambda/2) - 1]}{[4(\gamma - 1) + \lambda^2]\cosh(\lambda/2) - 4(\gamma - 1)}. \tag{23}$$

For each expression in (22) and (23) we may find the locus in H–γ space where $\delta = 10^{-1}$, 10^{-2} and 10^{-3}, by evaluating δ for a wide range of H and γ values and drawing a contour plot of the appropriate values of δ. The results of this simple process are shown in Figs. 1 and 2.

Each figure shows that the locus of values of δ, when δ is small, is approximately a straight line on a log–log graph. It is a straightforward matter to determine an asymptotic expansion of Eqs. 22 and 23 for large values of λ (see (20)); we quote the following expressions,
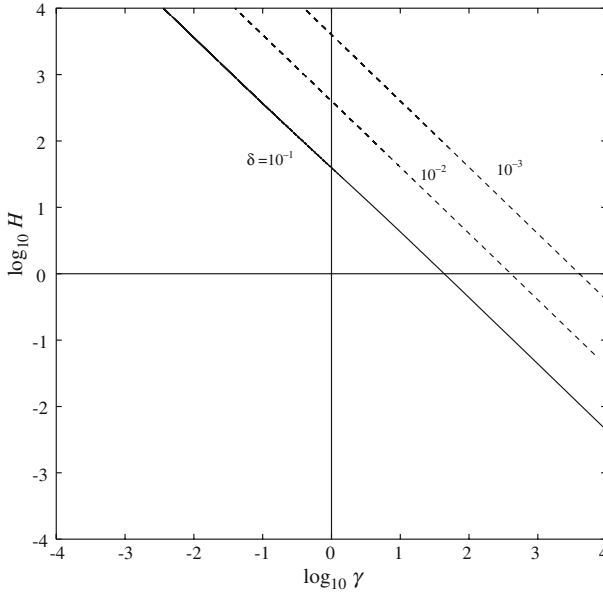
Fig. 1 Internal heating in the fluid phase ($\zeta = 0$): contours in H–γ space corresponding to $\delta = 10^{-1}$, 10^{-2} and 10^{-3}
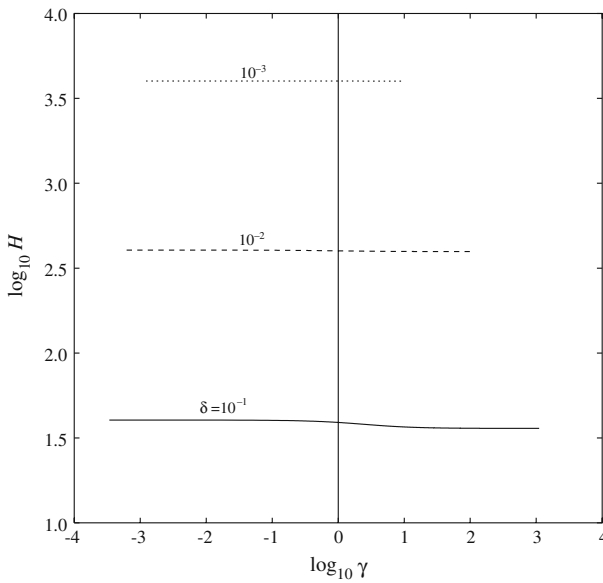

Fig. 2 Internal heating in the solid phase ($\zeta = \pi/2$): contours in H–γ space corresponding to $\delta = 10^{-1}$, 10^{-2} and 10^{-3}

$$H\gamma \sim \frac{4}{\delta} + 4\frac{(\gamma - 1)}{(\gamma + 1)} \quad \text{for} \quad \zeta = 0, \tag{24}$$

and

$$H \sim \frac{4}{\delta} - 4\frac{(\gamma - 1)}{(\gamma + 1)}, \quad \text{for} \quad \zeta = \pi/2. \tag{25}$$

Both these expressions are very highly accurate. Therefore, if we were to decide that LTE corresponds to $\delta \le 10^{-2}$, i.e. an error of less than 0.5%, then we would require

$$H\gamma > 400 + 4\frac{(\gamma - 1)}{(\gamma + 1)} \quad \text{for} \quad \zeta = 0, \tag{26}$$

and

$$H > 400 - 4\frac{(\gamma - 1)}{(\gamma + 1)}, \quad \text{for} \quad \zeta = \pi/2, \tag{27}$$

to guarantee LTE. In terms of the values of h, ϵ, k_{f} and k_{s}, the expressions corresponding to (24) and (25) are,

$$\frac{hL^2}{(1 - \epsilon)k_{\mathrm{s}}} > \frac{4}{\delta} + 4\frac{\epsilon k_{\mathrm{f}} - (1 - \epsilon)k_{\mathrm{s}}}{\epsilon k_{\mathrm{f}} + (1 - \epsilon)k_{\mathrm{s}}}, \quad \text{for} \quad \zeta = 0, \tag{28}$$

$$\frac{hL^2}{\epsilon k_{\mathrm{f}}} > \frac{4}{\delta} + 4\frac{(1 - \epsilon)k_{\mathrm{s}} - \epsilon k_{\mathrm{f}}}{(1 - \epsilon)k_{\mathrm{s}} + \epsilon k_{\mathrm{f}}}, \quad \text{for} \quad \zeta = \pi/2, \tag{29}$$

which shows a much stronger degree of symmetry between the two heating cases than is suggested by Eqs. 24 and 25.

4 Conclusions

In this short paper we have determined exact solutions for the temperature profiles in a porous channel with internal heating subject to local thermal non-equilibrium between the phases. From this it has been possible to provide very accurate conditions which must be satisfied to allow local thermal equilibrium to be assumed.

Should it now be deemed necessary to find the flow induced by the thermal profiles derived above, then this may be found easily by a suitable integration of the θ profile with account being taken of the direction of gravity. When heating is from below, then there remains the possibility of thermo-convective instability, a topic which we plan to study in a future paper.

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