UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2023
Unit Title: Modelling Techniques 2		Unit Code: ME20021
Year: 2022/23	Question Number: 3.	Page 1 of 1

Part		Mark

(a) Separation of variables: let $\theta = T(t)\cos n\pi x$ for $n = 0, 1, 2, \cdots$. Hence $T' = -\alpha n^2\pi^2 T$.

When $n = 0$ we have $T = A$. When $n \neq 0$ we have $T = Ae^{-\alpha n^2\pi^2 t}$.

Reconstructing θ and superposing yields (where 7 marks is allocated if A_0 is missing),

$$\theta = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\alpha n^2\pi^2 t}\cos n\pi x.$$

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(b) At $t = 0$ we have $\theta = x - x^2$, hence $x - x^2 = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x$, where

$$A_n = 2\int_0^1 (x - x^2)\cos n\pi x\, dx.$$

After integration we obtain $A_n = 0$ when n is odd, and $A_n = -4/n^2\pi^2$ when n is even. When $n = 0$ we have

$$A_0 = 2\int_0^1 (x - x^2)\, dx = \tfrac{1}{3}.$$

Hence the full solution is

$$\theta = \tfrac{1}{6} - \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} 4\frac{\cos n\pi x}{n^2\pi^2} e^{-\alpha n^2\pi^2 t}.$$

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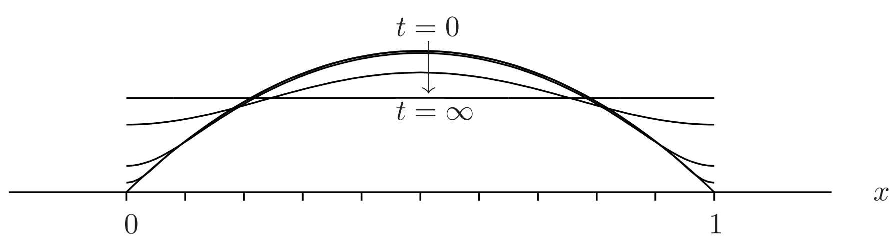
(c) The sketch is as below.



All curves must have zero slope at $x = 0$ and $x = 1$ for full marks.

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(d) The value $\frac{1}{2}A_0$ is the mean initial temperature or the long-term temperature which is achieved.

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Total	33

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Part		Mark
(a)	This is bookwork. $\mathcal{F}_s[f''] = \int_0^\infty f'' \sin \omega x \, dx =$ $= \Big[f'\Big]\Big[\sin \omega x\Big]_0^\infty - \Big[f\Big]\Big[\omega \cos \omega x\Big]_0^\infty + \int_0^\infty \Big[f\Big]\Big[-\omega^2 \sin \omega x\Big]\, dx$ $= \omega f(0) - \omega^2 F_s(\omega).$	5
(b)	The Fourier sine transform of $g(x)$ is $\mathcal{F}_s[g(x)] = \int_0^\infty g(x) \sin(\omega x)\, dx = \int_0^1 1 \sin(\omega x)\, dx = \dfrac{1-\cos\omega}{\omega}.$	8
(b)	The application of the Fourier Sine Transform to Fourier's equation (together with the zero boundary condition at $x = 0$) gives $\dfrac{\partial \Phi_s}{\partial t} = -\alpha\omega^2 \Phi_s.$ The general solution is, $\Phi_s = A(\omega)e^{-\alpha\omega^2 t}$	8
	At $t = 0$ we have $\phi = g(x)$, and hence $\Phi_s = (1-\cos\omega)/\omega$. Therefore $A = (1-\cos\omega)/\omega$.	2
	The solution for Φ_s is therefore $\Phi_s = \dfrac{(1-\cos\omega)}{\omega}e^{-\alpha\omega^2 t}$.	5
	Applying the inverse Fourier Sine Transform gives $\phi = \dfrac{2}{\pi}\int_0^\infty \dfrac{(1-\cos\omega)}{\omega}e^{-\alpha\omega^2 t} \sin\, \omega x \, d\omega.$	5

Total	33