3. Unsteady one–dimensional heat conduction takes place in a bar of unit length and the evolving temperature profile is governed by Fourier's equation,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2},$$

where θ is the temperature and α is the thermal diffusivity.

(a) If the ends of the bar are insulated, i.e. that $\partial\theta/\partial x$ is set to be zero at both $x = 0$ and $x = 1$, then use the technique of separation of variables to show that physically meaningful solutions may be written in the form of an infinite sum of terms. [10 marks]

(b) The initial temperature profile is

$$\theta = x(1-x) \qquad \text{at} \qquad t = 0.$$

Use the result of part (a) and the definition of the Fourier Cosine Series given below to determine the temperature within the bar. [15 marks]

(c) Provide a rough sketch to indicate how the temperature profile evolves in time. [5 marks]

(d) What is the significance of the value, $\frac{1}{2}A_0$? [3 marks]

You may use the following expression for the Fourier Cosine Series of a function, $f(x)$, in the range, $0 \le x \le 1$:

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x$$

where

$$A_n = 2\int_0^1 f(x)\cos n\pi x \, dx \qquad n = 0, \cdots, \infty.$$

4. The Fourier Sine Transform of $f(x)$ is given by $\mathcal{F}_s[f(x)] = F_s(\omega)$, where the definition of the Fourier Sine Transform is given at the end of the question.

(a) Show that

$$\mathcal{F}_s\left[\frac{d^2 f}{dx^2}\right] = \omega f(0) - \omega^2 F_s(\omega)$$

provided that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$. [5 marks]

(b) The function, $g(x)$, is a unit pulse and is defined as follows:

$$g(x,0) = \begin{cases} 1 & (0 \le x < 1) \\ 0 & (1 < x < \infty). \end{cases}$$

Show also that

$$\mathcal{F}_s\left[g(x)\right] = \frac{1-\cos\omega}{\omega}.$$

[8 marks]

(c) The conduction of heat in a semi-infinite slab is modelled by Fourier's equation

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}, \qquad x \ge 0, \quad t \ge 0,$$

where α is the thermal diffusivity and ϕ is the temperature. The initial temperature profile is given by $\psi(x,0) = g(x)$, where $g(x)$ is as given in Part (b), while the left hand boundary is maintained at a zero temperature, $\phi(0,t) = 0$.

If $\Phi_s(\omega,t)$ is the Fourier sine transform of $\phi(x,t)$ with respect to x, then determine $\Phi_s(\omega,t)$ by first taking the Fourier sine tranform of Fourier's equation. Then write down an expression for $\phi(x,t)$ in terms of an integral. [20 marks]

You may use the following expressions for the Fourier sine transform and its inverse:

$$\mathcal{F}_s[f(x)] = \int_0^\infty f(x)\,\sin(\omega x)\,dx \equiv F_s(\omega),$$

$$\mathcal{F}_s^{-1}[F_s(\omega)] = \frac{2}{\pi}\int_0^\infty F_s(\omega)\,\sin(\omega x)\,d\omega = f(x).$$