3. Unsteady one-dimensional heat conduction takes place in a bar of unit length and the evolving temperature field is governed by Fourier's equation,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2},$$

where θ is the temperature and α is the thermal diffusivity.

(a) The ends of the rod are located at $x = 0$ and $x = 1$, and they are held at $\theta = 0$ indefinitely. Use the technique of separation of variables to show that physically meaningful solutions may be written in the form of a suitable infinite sum of terms. [10 marks]

(b) At the initial time, $t = 0$, the temperature profile is given by,

$$\theta = \begin{cases} 1 - 2x & (0 \le x \le \frac{1}{2}) \\ 0 & (\frac{1}{2} \le x \le 1) \end{cases} .$$

Use the result of part (a) and the definition of the Fourier Sine Series given below to determine an expression for the subsequent evolution of the temperature field. [15 marks]

(c) On the same diagram, sketch both the initial temperature profile, as given above, and a profile that corresponds to the very initial stages of its evolution in time. [8 marks]

You may use the following expression for the half-range Fourier Sine Series of a function, $f(x)$, in the range, $0 \le x \le 1$:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

where

$$B_n = 2 \int_0^1 f(x) \sin(n\pi x)\, dx \qquad n = 1, \cdots, \infty.$$

4. The Fourier Sine Transform of $f(x)$ is given by $\mathcal{F}_s[f(x)] = F_s(\omega)$, where the definition of the Fourier Sine Transform is given at the end of the question.

(a) Show that

$$\mathcal{F}_s\left[\frac{d^2 f}{dx^2}\right] = \omega f(0) - \omega^2 F_s(\omega),$$

provided that $f(x) \longrightarrow 0$ and $f'(x) \longrightarrow 0$ as $x \longrightarrow \infty$. [8 marks]

(b) The displacement, $y(x,t)$, of a taut elastic string satisfies the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where c is the wave speed. The string lies in the region, $0 \leq x < \infty$ and it is assumed to be at rest (i.e. $y = 0$) for $t < 0$.

Suppose now that, at $t = 0$, the end of the string at $x = 0$ is moved instantaneously to the new displacement, $y = 1$, and is held there for all time.

Use the Fourier Sine Transform and the result from part (a) to show that the subsequent evolution of the string is given by,

$$y = \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos c\omega t)}{\omega} \sin \omega t \, d\omega.$$ [15 marks]

(c) Using other methods it is possible to show that the solution may also be written down in the very simple form, $y = H(ct - x)$, where H is the unit step function. Find the Fourier Sine Transform of $H(ct - x)$ in order to confirm that the solution to part (b) is correct. [5 marks]

What is the physical interpretation of this step-function solution? [5 marks]

You may use the following expressions for the Fourier sine transform and its inverse:

$$\mathcal{F}_s[f(x)] = \int_0^\infty f(x) \, \sin(\omega x) \, dx \equiv F_s(\omega),$$

$$\mathcal{F}_s^{-1}[F_s(\omega)] = \frac{2}{\pi} \int_0^\infty F_s(\omega) \, \sin(\omega x) \, d\omega = f(x).$$

DNJ/DASR