2. The unsteady pressure distribution in a clarinet is like that of a uniform pipe which is open at one end and closed at the other. The pressure satisfies the wave equation,

$$\frac{\partial^2 P}{\partial t^2} = c^2 \frac{\partial^2 P}{\partial x^2},$$

where P is the gauge pressure and c is the wave speed. Additionally, the pressure satisfies $P = 0$ at the open end at $x = 0$ and $\partial P/\partial x = 0$ at the closed end at $x = 1$.

(a) Use the technique of separation of variables to show that physically meaningful solutions may be written in the form of a suitable infinite sum of terms. [10 marks]

(b) At an initial time, $t = 0$, the pressure distribution satisfies

$$P = x \quad \text{and} \quad \frac{\partial P}{\partial t} = 0.$$

Use the result of part (a) and the definition of the quarter-range Fourier Sine Series given below to determine the subsequent evolution of the pressure field. [15 marks]

You may use the following expression for the quarter-range Fourier Sine Series of a function, $f(x)$, in the range, $0 \le x \le 1$:

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} B_n \sin(n\pi x/2)$$

where

$$B_n = 2\int_0^1 f(x)\sin(n\pi x/2)\,dx \qquad n = 1, \cdots, \infty.$$

3. The Fourier Cosine Transform of $f(x)$ is given by $\mathcal{F}_c[f(x)] = F_c(\omega)$, where the definition of the Fourier Cosine Transform is given at the end of the question.

(a) Show that

$$\mathcal{F}_c\left[\frac{d^2 f}{dx^2}\right] = -f'(0) - \omega^2 F_c(\omega)$$

provided that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$. [5 marks]

(b) The function, $g(x)$, is a triangular pulse which is defined as follows:

$$g(x,0) = \begin{cases} 1-x & (0 \leq x < 1) \\ 0 & (1 < x < \infty). \end{cases}$$

Find its Fourier Cosine Transform. [5 marks]

(c) The conduction of heat in a semi-infinite slab is modelled by Fourier's equation

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}, \qquad x \geq 0, \quad t \geq 0,$$

where α is the thermal diffusivity and θ is the temperature. The initial temperature profile at $t = 0$ is given by $\theta(x, 0) = g(x)$, where $g(x)$ is given in Part (b), while the left hand boundary is insulated, i.e. where $\partial\theta/\partial x = 0$ at $x = 0$.

If $\Theta_c(\omega, t)$ is defined to be the Fourier Cosine Transform of $\theta(x, t)$ with respect to x, then determine $\Theta_c(\omega, t)$ by first taking the Fourier Cosine Tranform of Fourier's equation. Then write down an expression for $\theta(x, t)$ in terms of an integral. [10 marks]

(d) Give a sketch of how you would expect the temperature profile to evolve in time. [5 marks]

You may use the following expressions for the Fourier Cosine Transform and its inverse:

$$\mathcal{F}_c[f(x)] = \int_0^\infty f(x)\,\cos(\omega x)\,dx \equiv F_c(\omega),$$

$$\mathcal{F}_c^{-1}[F_c(\omega)] = \frac{2}{\pi}\int_0^\infty F_c(\omega)\,\cos(\omega x)\,d\omega = f(x).$$

DASR