3. Steady two–dimensional heat conduction occurs in a bar of unit width in the x-direction, but which is semi-infinite in the y-direction (i.e. $0 \leq y < \infty$). The temperature field is governed by Laplace's equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

where T is the temperature.

(a) If T is set to be zero at both $x = 0$ and $x = 1$, then use the technique of separation of variables to show that physically meaningful solutions may be written in the form of an infinite sum of terms. [8 marks]

(b) The temperature profile at $y = 0$ is given by

$$T = x - x^3.$$

Use the result of part (a) and the definition of the Fourier Sine Series given below to show that the temperature within the bar is given by

$$T(x,t) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{\pi^3 n^3} e^{-n\pi y} \sin n\pi x.$$ [12 marks]

(c) How would the solution given in part (b) be modified for the case where the solid bar is semi-infinite in the x-direction, has a unit width in the y-direction, where $T = 0$ on both $y = 0$ and $y = 1$, and where $T = y - y^3$ on $x = 0$? [5 marks]

(d) Suppose that the semi-infinite bar described at the start of this question now has a unit length in the y-direction instead. There is a new boundary condition that $T = 0$ at $y = 1$. Modify the solution which was obtained in parts (a) and (b) to find the temperature distribution in this square region. [8 marks]

You may use the following expression for the Fourier Sine Series of a function, $f(x)$, in the range, $0 \leq x \leq 1$:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

where

$$B_n = 2\int_0^1 f(x) \sin n\pi x \, dx \qquad n = 1, \cdots, \infty.$$

4. The Fourier transform, $F(\omega)$, of the function $f(x)$ is defined by

$$\mathcal{F}\left[f(x)\right] = F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx.$$

(a) Show that

$$\mathcal{F}\left[\frac{d^2 f}{dx^2}\right] = -\omega^2 F(\omega),$$

assuming that both $f(x)$ and $f'(x)$ are asymptotically zero as $x \longrightarrow \pm\infty$. [8 marks]

(b) Show that

$$\mathcal{F}\left[f(x+a)\right] = e^{aj\omega} F(\omega).$$

[5 marks]

(c) Wave motion of an infinitely long taut string is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where c is the wavespeed. If $U(\omega, t)$ is the Fourier transform of $u(x,t)$ with respect to x show that U satisfies the equation

$$\frac{\partial^2 U}{\partial t^2} + \omega^2 c^2 U = 0.$$

[5 marks]

(d) Given the initial displacement and velocity for u,

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0,$$

solve the above equation for U in terms of $F(\omega)$, the Fourier transform of $f(x)$. Use the result given in part (b) to obtain an explicit expression for the resulting motion of the string, $u(x,t)$. [10 marks]

[You may need to use the identity $\cos\alpha = (e^{j\alpha} + e^{-j\alpha})/2$.]

Describe in detail the physical meaning of the different parts of your solution. [5 marks]

DNJ/DASR