**UNIVERSITY OF BATH**
**DEPARTMENT OF MECHANICAL ENGINEERING**

Outline Solution to Examination Question

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|---|---|---|
| Examiner: Dr D A S Rees | | Date May 2019 |
| Unit Title: Modelling Techniques 2 | | Unit Code: ME20021 |
| Year: 2018/19 | Question Number: 3. | Page 1 of 1 |

| Part | | Mark |
|---|---|---|

(a) The boundary condition, $T=\theta^2$, is even, so we only need cosines and the constant term. Therefore let, $T(r,\theta)=R(r)\cos n\theta$.

Hence $r^2R''+rR'-n^2R=0$, a Cauchy-Euler equation, for which solutions take the form, $R=r^p$. Substitution yields,

$$\Big[p(p-1)+p-n^2\Big]r^p=0 \qquad \Rightarrow \qquad p^2=n^2 \qquad \Rightarrow \qquad p=\pm n.$$

Therefore $R=Ar^{-n}+Br^n$, when $n\neq 0$ and hence a fundamental solution is,

$$T=\Big[A_nr^{-n}+B_nr^n\Big]\cos n\theta, \qquad (n=1,2,\cdots).$$

When $n=0$ we have $r^2R''+rR'=0$. Division by $r$ and a slight rearrangement gives, $(rR')'=0$. Integration gives $R'=B/r$. Further integration gives $R=A+B\ln r$. We may add all of these solutions to give,

$$T=A_0+B_0\ln r+\sum_{n=1}^{\infty}\Big[A_nr^{-n}+B_nr^n\Big]\cos n\theta. \tag{1}$$

We need $B_0=0$ and $B_n=0$ in Eq. (1) to avoid singularities at $r=0$. Hence

$$T=\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_nr^n\cos n\theta,$$

where the usual $\frac{1}{2}$ has been used as the coefficient of $A_0$. 15

At $r=1$ we have $T=\theta^2$, hence $\theta^2=\frac{1}{2}A_0\sum_{n=1}^{\infty}A_n\cos n\theta$, where

$$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\theta^2\,\cos n\theta=\frac{2}{\pi}\int_0^{\pi}\theta^2\,\cos n\theta=\frac{4}{n^2}(-1)^n,$$

after integration by parts. Also $A_0=\frac{2}{3}\pi^2$. Hence the required solution is,

$$T=\tfrac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^nr^n\cos n\theta.$$

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(b) For the region occupying the exterior of a unit circle then Eq. (1) still applies, but now we retain terms which do not grow as $r\to\infty$. As the boundary condition at $r=1$ is the same as for part (b) the same Fourier series analysis applies. Hence the solution is

$$T=\tfrac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^nr^{-n}\cos n\theta.$$

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|---|---|
| Total | 33 |

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| Part | | Mark |
|---|---|---|
| (a) | This is bookwork. $\mathcal{F}_s[f''] = \int_0^\infty f'' \sin\omega x \, dx =$ $= \Big[f'\Big]\Big[\sin\omega x\Big]_0^\infty - \Big[f\Big]\Big[\omega\cos\omega x\Big]_0^\infty + \int_0^\infty \Big[f\Big]\Big[-\omega^2\sin\omega x\Big]\,dx$ $= \omega f(0) - \omega^2 F_s(\omega).$ | 5 |
| (b) | The Fourier sine transform of $g(x)$ is $\mathcal{F}_s[g(x)] = \int_0^\infty g(x)\sin(\omega x)\,dx = \int_0^1 1\,\sin(\omega x)\,dx = \dfrac{1-\cos\omega}{\omega}.$ | 8 |
| (b) | The application of the Fourier Sine Transform to Fourier's equation (together with the zero boundary condition at $x=0$) gives $\dfrac{\partial\Phi_s}{\partial t} = -\alpha\omega^2\Phi_s.$ The general solution is, $\Phi_s = A(\omega)e^{-\alpha\omega^2 t}$ | 8 |
| | At $t=0$ we have $\phi = g(x)$, and hence $\Phi_s = (1-\cos\omega)/\omega$. Therefore $A = (1-\cos\omega)/\omega$. | 2 |
| | The solution for $\Phi_s$ is therefore $\Phi_s = \dfrac{(1-\cos\omega)}{\omega}e^{-\alpha\omega^2 t}$. | 5 |
| | Applying the inverse Fourier Sine Transform gives $\phi = \dfrac{2}{\pi}\int_0^\infty \dfrac{(1-\cos\omega)}{\omega}e^{-\alpha\omega^2 t}\sin\,\omega x\,d\omega.$ | 5 |
| | Total | 33 |