3. Two–dimensional steady-state heat conduction takes place within a circular cylinder of unit radius. The temperature field is governed by Laplace's equation in polar coordinates:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0,$$

where T is the temperature, r the radial coordinate and θ the tangential or angular coordinate.

(a) The temperature of the external boundary of the cylinder at $r = 1$ has the profile,

$$T = \theta^2, \quad (-\pi < \theta < \pi).$$

Use the technique of separation of variables and the definition of the Fourier Series given below to find an expression for the temperature field inside the cylinder in the form of an infinite sum of terms. [25 marks]

(b) Use the solution obtained in part (a) to write down the corresponding solution for the case when $r = 1$ represents the *inner* boundary of a solid lying in the range, $1 \leq r < \infty$, and where the temperature at $r = 1$ has the same profile as for part (a). [8 marks]

You may use the following expression for the Fourier Series of a function, $f(\theta)$, in the range, $-\pi \leq \theta \leq \pi$:

$$f(\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \cos n\theta + B_n \sin n\theta\Big],$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta \qquad n = 0, \cdots, \infty,$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta \qquad n = 1, \cdots, \infty.$$

4. The Fourier Sine Transform of $f(x)$ is given by $\mathcal{F}_s[f(x)] = F_s(\omega)$, where the definition of the Fourier Sine Transform is given at the end of the question.

(a) Show that

$$\mathcal{F}_s\left[\frac{d^2 f}{dx^2}\right] = \omega f(0) - \omega^2 F_s(\omega)$$

provided that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$. [5 marks]

(b) The function, $g(x)$, is a unit pulse and is defined as follows:

$$g(x,0) = \begin{cases} 1 & (0 \le x < 1) \\ 0 & (1 < x < \infty). \end{cases}$$

Show also that

$$\mathcal{F}_s\left[g(x)\right] = \frac{1-\cos\omega}{\omega}.$$

[8 marks]

(c) The conduction of heat in a semi-infinite slab is modelled by Fourier's equation

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}, \qquad x \ge 0, \quad t \ge 0,$$

where α is the thermal diffusivity and ϕ is the temperature. The initial temperature profile is given by $\psi(x,0) = g(x)$, where $g(x)$ is as given in Part (b), while the left hand boundary is maintained at a zero temperature, $\phi(0,t) = 0$.

If $\Phi_s(\omega, t)$ is the Fourier sine transform of $\phi(x,t)$ with respect to x, then determine $\Phi_s(\omega,t)$ by first taking the Fourier sine tranform of Fourier's equation. Then write down an expression for $\phi(x,t)$ in terms of an integral. [20 marks]

You may use the following expressions for the Fourier sine transform and its inverse:

$$\mathcal{F}_s[f(x)] = \int_0^\infty f(x)\,\sin(\omega x)\,dx \equiv F_s(\omega),$$

$$\mathcal{F}_s^{-1}[F_s(\omega)] = \frac{2}{\pi}\int_0^\infty F_s(\omega)\,\sin(\omega x)\,d\omega = f(x).$$