3. Unsteady one–dimensional heat conduction takes place in a bar of unit length and the evolving temperature profile is governed by Fourier's equation,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2},$$

where θ is the temperature and α is the thermal diffusivity.

(a) The bar is held at a zero temperature at $x = 0$, and the boundary at $x = 1$ is insulated (i.e. $\partial\theta/\partial x = 0$). Use the technique of separation of variables to show that physically meaningful solutions may be written in the form of a suitable infinite sum of terms. [13 marks]

(b) The initial temperature profile is

$$\theta = 2x - x^2 \qquad \text{at} \qquad t = 0.$$

Use the result of part (a) and the definition of the quarter-range Fourier Sine Series given below to show that the temperature within the bar is given by

$$\theta(x,t) = \sum_{\substack{n=1 \\ n \,\mathrm{odd}}}^{\infty} \frac{32}{\pi^3 n^3}\, e^{-\alpha n^2 \pi^2 t/4} \sin(n\pi x/2).$$

[20 marks]

You may use the following expression for the quarter-range Fourier Sine Series of a function, $f(x)$, in the range, $0 \le x \le 1$:

$$f(x) = \sum_{\substack{n=1 \\ n \,\mathrm{odd}}}^{\infty} B_n \sin(n\pi x/2)$$

where

$$B_n = 2\int_0^1 f(x)\sin(n\pi x/2)\,dx \qquad n = 1, 3, 5, \cdots, \infty.$$

4. The Fourier transform, $F(\omega)$, of the function $f(x)$ is defined by

$$\mathcal{F}[f(x)] = F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx.$$

(a) Show that

$$\mathcal{F}\left[\frac{d^2 f}{dx^2}\right] = -\omega^2 F(\omega)$$

assuming that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \pm\infty$. [5 marks]

(b) Find $\mathcal{F}[e^{-a|x|}]$ (where $a > 0$) and hence use the Symmetry Theorem (stated below) to show that

$$\mathcal{F}\left[\frac{1}{\pi}\,\frac{a}{a^2+x^2}\right] = e^{-a|\omega|}.$$

[8 marks]

(c) The steady conduction of heat in two dimensions is governed by Laplace's equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0,$$

where θ is the temperature. If $\Theta(\omega, y)$ is the Fourier Transform of $\theta(x, y)$ with respect to x, show that Θ satisfies the equation

$$\frac{\partial^2\Theta}{\partial y^2} - \omega^2\Theta = 0.$$

[5 marks]

(d) A solid conducting block occupies the region $-\infty < x < \infty$, $y \geq 0$. A point source of heat is applied at the origin and is such that the boundary temperature at $y = 0$ is given by

$$\theta = \delta(x),$$

where $\delta(x)$ is the unit impulse.

Use all the above results to determine the temperature distribution in the solid. Hence show that the $\theta = \frac{1}{2\pi}$ contour lies on a circle of radius 1 which is centred at $(x, y) = (0, 1)$. [15 marks]

Symmetry Theorem: $\mathcal{F}[f(x)] = F(\omega) \quad \Rightarrow \quad \mathcal{F}[F(x)] = 2\pi f(-\omega)$.