Question 3

Unsteady one–dimensional heat conduction takes place in a bar of unit length and the evolving temperature profile is governed by Fourier's equation,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2},$$

where θ is the temperature and α is the thermal diffusivity.

(a) If the ends of the bar are insulated, i.e. that $\partial\theta/\partial x$ is set to be zero at both $x = 0$ and $x = 1$, then use the technique of separation of variables to show that physically meaningful solutions may be written in the form of an infinite sum of terms. **[10 marks]**

(b) The initial temperature profile is

$$\theta = x(1 - x) \qquad \text{at} \qquad t = 0.$$

Use the result of part (a) and the definition of the Fourier Cosine Series given below to determine the temperature within the bar. **[16 marks]**

(c) Provide a rough sketch to indicate how the temperature profile evolves with time. **[5 marks]**

(d) What is the significance of the value, $\frac{1}{2}A_0$? **[2 marks]**

You may use the following expression for the Fourier Cosine Series of a function, $f(x)$, in the range, $0 \le x \le 1$:

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x$$

where

$$A_n = 2\int_0^1 f(x) \cos n\pi x \, dx \qquad n = 0, \cdots, \infty.$$

Question 4

(a) For the function, $f(x)$, show that

$$\mathcal{F}_s\left[\frac{d^2 f}{dx^2}\right] = \omega f(0) - \omega^2 F_s(\omega)$$

provided that both $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$. **[8 marks]**

(b) The displacement, $y(x,t)$, of a taut elastic string satisfies the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where c is the wave speed. The string lies in the region $0 \leq x < \infty$ and is assumed to be in equilibrium for $t < 0$, i.e. that it has zero displacement and velocity.

Suppose now that, at $t = 0$, the end of the string at $x = 0$ is moved instantaneously to the new displacement, $y = 1$, and is held there for all time. Use the Fourier Sine Transform and the result from part (a) to show that the subsequent displacement of the string is given by,

$$y = \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos c\omega t)}{\omega} \sin \omega x \, d\omega.$$ **[15 marks]**

(c) Using other methods it is possible to show that the solution may also be written down in a very simple form: $y = H(ct - x)$, where H is the unit step function. Find the Fourier Sine Transform with respect to x of this expression in order to confirm that the solution to part (b) quoted above is correct. **[5 marks]**

What is the physical interpretation of the step-function solution? **[5 marks]**

You may use the following expressions for the Fourier sine transform and its inverse:

$$\mathcal{F}_s[f(x)] = \int_0^\infty f(x) \sin(\omega x) \, dx \equiv F_s(\omega),$$

$$\mathcal{F}_s^{-1}[F_s(\omega)] = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin(\omega x) \, d\omega = f(x).$$