SOLUTION OF A SYSTEM OF EQUATIONS

This handout will summarise briefly how we may solve systems of homogeneous constant-coefficient ODEs. We'll take as the example the following pair:

$$y' = -3y - z, \qquad z' = -3z - y, \tag{1a, b}$$

and they will be solved subject to $y(0) = 2$ and $z(0) = 0$. There are various ways in which this might be done.

Method 1.

We may use Eq. (1a) to write z in terms of y:

$$z = -3y - y'. \tag{2}$$

This and its derivative, which is $z' = -3y' - y''$, may now be substituted into the other equation, (1b), to obtain a second order equation for y. We get,

$$y'' + 6y' + 8y = 0. \tag{3}$$

This may be solved in the usual way by substituting $y = e^{\lambda t}$. The auxiliary equation is

$$\lambda^2 + 6\lambda + 8 = 0 \qquad \Rightarrow \qquad \lambda = -2, -4. \tag{4}$$

Therefore the solution for y is

$$y = Ae^{-2t} + Be^{-4t}, \tag{5}$$

where A and B are arbitrary constants. We may now find z by substituting for y into Eq. (2). This yields,

$$z = -Ae^{-2t} + Be^{-4t}. \tag{6}$$

We may also write the general solution which we have just found in a vector form:

$$\begin{pmatrix} y \\ z \end{pmatrix} = Ae^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{7}$$

Application of the initial conditions, $y(0) = 2$ and $z(0) = 0$, to Eqs. (5) and (6) give $A + B = 2$ and $B - A = 0$, respectively. Hence $A = B = 1$ and the final solution is

$$\begin{pmatrix} y \\ z \end{pmatrix} = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{8}$$

or, alternatively,

$$y = e^{-2t} + e^{-4t}, \qquad z = -e^{-2t} + e^{4t}. \tag{9}$$

Method 2.

Method 1 showed clearly that systems of linear constant-coefficient equations may also be solved using the $e^{\lambda t}$ substitution. We will therefore do that directly to the equations thenselves: let

$$y(t) = Ye^{\lambda t}, \qquad z(t) = Ze^{\lambda t}, \tag{10a, b}$$

where Y and Z are constants. Although they are both arbitrary, they are related to one another as Eq. (7) implies. Substitution of (10) into (1) yields,

$$\lambda Y = -3Y - Z, \qquad \lambda Z = -3Z - Y. \tag{11a, b}$$

We may eliminate either Y or Z from between this pair of simultaneous equations. Assuming that we have eliminated Z, we get

$$(\lambda + 3)^2 Y = Y \qquad \Rightarrow \qquad (\lambda + 3)^2 = 1 \qquad \Rightarrow \qquad \lambda = -2, -4. \tag{12}$$

These are, of course, the same values of λ that Method 1 gave, and they are also identical to what happens when we eliminate Y instead. Therefore we may write

$$y = Ae^{-2t} + Be^{-4t}, \tag{13}$$

using two arbitrary constants, A and B. This solution is comprised of two components of the form given in (10a) where we have set $Y = A$ when $\lambda = -2$, and $Y = B$ when $\lambda = -4$. Equations (11a) and (11b) both show that the corresponding values for Z for the respective two values of λ are $Z = -A$ and $Z = B$. Therefore the solution for z is,

$$z = -Ae^{-2t} + Be^{-4t}. \tag{14}$$

Equation (14) could also be obtained by substituting (13) directly into Eq. (1a).

Having now acquired the same general solution as Method 1, the problem is completed by applying the initial conditions, and this results in Eqs. (8) and (9).

Comments

This overall solution may also be accomplished using eigenvalue and eigenvector theory, which is to follow later in the unit. The following forms a link to that theory.

Note that Eqs. (11a) and (11b) may be written in the matrix/vector form,

$$\begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \lambda \begin{pmatrix} Y \\ Z \end{pmatrix}, \tag{15}$$

which is the classical eigenvalue problem. The eigenvalues turn out to be, $\lambda = -2, -4$, which isn't a surprise, and the respective eigenvectors are

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{16}$$

All this information may be combined to give,

$$\begin{pmatrix} y \\ z \end{pmatrix} = Ae^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{17}$$

which is the same as Eq. (7),