Department of Mechanical Engineering, University of Bath

Mathematics 2 ME10305

Problem Sheet — Root finding and iteration schemes.

Q1. The aim for this question is to repeat some of the techniques used in the lectures to find roots of equations.

(a) Use a suitable sketch to find the number of roots there are likely to be of the equation, $f(x) = x^3 - 2x + 1 = 0$.

(b) Use two *ad hoc* iteration schemes to determine the roots of the equation.

(c) Use the Newton–Raphson scheme with $x_0 = 1.1$ as the initial iterate to find one of those roots.

(d) Taking this root, use the perturbation method to determine how quickly each method used converges to that root.

Q2. Find the only real root of the cubic $x^3 - x^2 - x - 1 = 0$ correct to six significant figures. Use any method you like.

Q3. Use a suitable sketch to show that $f(x) = e^{-x} - x = 0$ has only one root. Use both the possible ad hoc schemes and the Newton-Raphson method to find that root. Analyze the approach to the solution for all three methods by setting $x_n = X + \epsilon_n$ where X is the solution of $f(X) = 0$, i.e. it satisfies $e^{-X} = X$.

Q4. Use the Newton-Raphson method to find the first 4 positive roots of $f(x) = x \sin x - 1 = 0$. Rough locations of the roots may be obtained using a suitable sketch.

Q5. So let us create a general perturbation analysis of the convergence of the Newton-Raphson method towards a double root. We'll fix the roots to be at $x = 0$ reflects a general situation perfectly, and therefore we will consider $f(x) = x^2 g(x)$ where $g(0) \neq 0$. Write down the Newton-Raphson formula for this $f(x)$, and use a perturbation analysis to determine how quickly the iteration scheme will converge to $x = 0$. What happens when we have $f(x) = x^m g(x)$ where m is a positive integer?

Q6. [This question is best tackled using some suitable software to undertake the computations.]

The objective is to find the zeros of the function, $f(x) = x^{1/3} - \ln x$, where it is no secret that any such zeros must be positive. Use both of the possible *ad hoc* methods and the Newton-Raphson method to find these zeros. I am not sure that it will be useful to sketch this function, but trialling a few tentative values of x is a good start.

Q7. [If you have access to a machine/software which can compute with complex numbers, then you may undertake this question, should you wish.]

Write down the Newton-Raphson scheme for $f(x) = x^2 + 1$. Now use $x_0 = 0.5 + 0.5j$ as the initial iterate. To what value does the Newton-Raphson scheme converge?

Use the same method for finding the square root of $2j$. Use $x_0 = 1 + 0j$ in this case.

Q8. [This is a project-style of question. It is lengthy and intricate, but it ends up with an algebraic equation to solve for which the Newton-Raphson method is well-suited. The background application is on the vibrations of a beam.]

First, an introduction to Ordinary Differential Eigenvalue problems. I'll summarise the process first with a 2nd order ODE, and your job will be to apply the same ideas to a 4th order ODE.

The vibrations of a taut string are described by the wave equation, and eventually one obtains the ODE,

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \qquad \text{subject to } \; y(0) = 0, \; y(1) = 0.$$

The value, y, is a displacement, like that of a violin string, and the boundary conditions represent a zero displacement at both ends, which is what one expects of a violin. Clearly $y = 0$ satisfies the ODE and boundary conditions, but we've heard violins and therefore we need nonzero solutions. The value, ω, is related to the frequency of vibration of the string, and nonzero solutions (eigensolutions!) arise for certain frequencies only, and it is these values which we seek (eigenvalues!). The analysis proceeds as follows.

The general solution is $y = A\cos\omega x + B\sin\omega x$. Given that $y(0) = 0$ we must therefore have $A = 0$, which means that we now have $y = B\sin\omega x$. Application of $y(1) = 0$ yields,

$$B\sin\omega = 0.$$

We can't have $B = 0$ because that means that string has no displacement, and that defeats the purpose of the analysis. So we must have $\omega = n\pi$, where n is a positive integer; these values of ω are called the eigenvalues of the ODE. For a chosen value of n, the associated disturbance shape is $y = B\sin n\pi x$ where B is arbitary; these are the eigensolutions.

Your task, should you wish to take it on, is to use a similar analysis of the corresponding equation for a beam, namely,

$$\frac{d^4y}{dx^4} - \omega^4 y = 0 \qquad \text{subject to } \; y(0) = y'(0) = 0, \; y(1) = y'(1) = 0.$$

The boundary conditions are consistent with those of a cantilever: zero displacement and zero slope.

(a) Use the substitution, $y = e^{\lambda x}$, to write down the general solution in terms of four functions and with four arbitrary constants. Where you have to choose between exponentials and hyperbolic functions, I would advise the hyperbolics on this occasion. Sorry.

b) Now apply the boundary conditions to get four algebraic equations. The following will be a somewhat arduous trek. The aim is to try to eliminate three of the arbitrary constants in order to have an equation involving the last arbitrary constant and an expression involing ω. If this has worked correctly you should get

$$\cos\omega\,\cosh\omega = 1. \tag{1}$$

(c) Sketch both $\cos\omega$ and $1/\cosh\omega$ to estimate where the first root of Eq. (1) might be. (Ignore the obvious one at $x = 0$ which actually yields nothing of any use!)

(d) Apply Newton-Raphson to find this first value of ω. Again, ω is the freqency of vibration of the beam and, given how much more constrained the beam is compared with the string, you should obtain a higher lowest frequency here. In the solutions I will provide the first four values of ω and the corresponding shapes of vibration.

D.A.S.R. 15/03/2021