

Solutions to Laplace Transforms Sheet 1

1. Find the Laplace Transforms of the following functions using the definition of the Laplace Transform (rather than by looking up the result in a table):

$$(a) \quad \mathcal{L}[e^{3t}] = \int_0^{\infty} e^{3t} e^{-st} dt = \int_0^{\infty} e^{-(s-3)t} dt = \frac{1}{3-s} \left[e^{-(s-3)t} \right]_0^{\infty} = \frac{1}{s-3}.$$

Note that this result relies on the fact that $s > 3$; when $s \leq 3$ the result is infinite and we say that the Laplace Transform does not exist in this case.

$$(b) \quad \mathcal{L}[e^{-3t}] = \frac{1}{s+3} \text{ by analogy with (a). Here we require } s > -3 \text{ for the transform to exist.}$$

$$(c) \quad \mathcal{L}[\cos \omega t] = \int_0^{\infty} (\cos \omega t) e^{-st} dt = \frac{s}{s^2 + \omega^2} \text{ using integration by parts. Here we require } s > 0 \text{ for the Transform to exist.}$$

$$(d) \quad \mathcal{L}[te^{-3t}] = \int_0^{\infty} te^{-(s+3)t} dt = \frac{1}{(s+3)^2} \text{ using integration by parts. Again } s > -3 \text{ for the Transform to exist.}$$

$$(e) \quad \mathcal{L}[t^3] = \frac{6}{s^4}.$$

- (f) $\mathcal{L}[t \cos \omega t]$ will, when written out as the full integral,

$$\int_0^{\infty} t \cos(\omega t) e^{-st} dt,$$

be found to have an integrand which is the product of three functions. This can be integrated but it is a very long process and a real hassle which involves finding other integrals. However, it may be integrated much more quickly by changing $\cos \omega t$ to $\exp j\omega t$. Then we combine the two exponentials, undertake the integration by parts and evaluating the result between the limits, and then taking the real part of the final answer. For the sake of brevity I'll miss out the integration by parts here but this process may be summarised as follows.

$$\begin{aligned} & \int_0^{\infty} t \cos(\omega t) e^{-st} dt \\ &= \operatorname{Re} \int_0^{\infty} t e^{j\omega t} e^{-st} dt = \operatorname{Re} \int_0^{\infty} t e^{-(s-j\omega)t} dt = \operatorname{Re} \frac{1}{(s-j\omega)^2} = \operatorname{Re} \frac{(s+j\omega)^2}{(s^2+\omega^2)^2} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}. \end{aligned}$$

An alternative way to do this is by first noting that

$$\int_0^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2 + \omega^2}.$$

Now differentiate both sides with respect to ω . The left and side gives,

$$\frac{\partial}{\partial \omega} \left[\int_0^{\infty} \sin(\omega t) e^{-st} dt \right] = \int_0^{\infty} t \cos(\omega t) e^{-st} dt.$$

The right hand side yields,

$$\frac{\partial}{\partial \omega} \left[\frac{\omega}{s^2 + \omega^2} \right] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}.$$

(g) $\mathcal{L}[f'''(t)] = -f''(0) - sf'(0) - s^2f(0) + s^3F(s)$, where $F(s) = \mathcal{L}[f(t)]$. Three integrations by parts for this one.

(h) $\mathcal{L}[f(t)] = (1 - e^{-s})/s$. Should be an easy integral!

(i) $\mathcal{L}[\cosh \omega t] = \frac{1}{2} \left(\frac{1}{s + \omega} + \frac{1}{s - \omega} \right)$. This follows quickly from $\cosh \omega t = \frac{1}{2}(e^{\omega t} + e^{-\omega t})$.

(j) $\mathcal{L}[t^2 e^{-t}]$ should come to $2/(s + 1)^3$.

(k) Let $I = \mathcal{L}[t^{-1/2}] = \int_0^{\infty} t^{-1/2} e^{-st} dt$. Now let $x = (st)^{1/2}$. Hence $dx = (s^{1/2}/2t^{1/2})dt$, or, $(2/s^{1/2})dx = t^{-1/2}dt$, and therefore the integral becomes,

$$I = \int_0^{\infty} \frac{2}{s^{1/2}} e^{-x^2} dx = \frac{2}{s^{1/2}} \int_0^{\infty} e^{-x^2} dx = \frac{2}{s^{1/2}} \frac{\pi^{1/2}}{2} = \frac{\pi^{1/2}}{s^{1/2}}$$

on using the given result that

$$\int_0^{\infty} e^{-x^2} dx = \frac{\pi^{1/2}}{2}.$$

2. Use the Laplace Transform to solve the following equations:

(a) $\frac{dy}{dt} + 4y = 6, \quad y(0) = 2.$

$\mathcal{L}[y'] = sY - y(0) = sY - 2$. Hence the ODE reduces to $(sY - 2) + 4Y = 6/s$, from which Y may be shown to be

$$Y = \frac{2(s + 3)}{s(s + 4)} \equiv \frac{A}{s} + \frac{B}{s + 4}.$$

Multiplying by $s(s + 4)$ we obtain $2s + 6 = A(s + 4) + Bs$. When $s = 0$ we see that $A = 3/2$, and when $s = -4$ we get $B = 1/2$. Hence

$$Y = \frac{1}{2} \left[\frac{3}{s} + \frac{1}{s + 4} \right] \quad \Rightarrow \quad y = \frac{1}{2} [3 + e^{-4t}].$$

$$(b) \quad \frac{d^2y}{dt^2} + 16y = 0, \quad y(0) = 0, \quad \frac{dy}{dt} = 1.$$

$\mathcal{L}[y''] = s^2Y - y'(0) - sy(0) = s^2Y - 1$. The equation reduces to

$$(s^2Y - 1) + 16Y = 0 \quad \Rightarrow \quad Y = \frac{1}{s^2 + 16} = \frac{1}{4} \left[\frac{4}{s^2 + 4^2} \right] \quad \Rightarrow \quad y = \frac{1}{4} \sin 4t$$

using tables.

$$(c) \quad \frac{d^2y}{dt^2} + 4y = 29e^{-5t}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = -3.$$

The equation becomes,

$$s^2Y + 3 + 4Y = \frac{29}{s + 5} \quad \Rightarrow \quad Y = \frac{29}{(s + 5)(s^2 + 4)} - \frac{3}{s^2 + 4}.$$

After some partial fractions work, this simplifies to,

$$Y = \frac{1}{s + 5} + \frac{2}{s^2 + 4} - \frac{s}{s^2 + 4}.$$

After using standard results for the LTs of exponentials and sinusoids, we obtain,

$$y = e^{-5t} + \sin 2t - \cos 2t.$$

$$(d) \quad y''' + y'' + 4y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 3, \quad y''(0) = -5.$$

Using the standard results for the LTs of the various derivatives, we get,

$$\left[s^3Y - y''(0) - sy'(0) - s^2y(0) \right] + \left[s^2Y - y'(0) - sy(0) \right] + 4 \left[sY - y(0) \right] + 4Y = 0.$$

Using the initial conditions and simplifying, we get,

$$(s^3 + s^2 + 4s + 4)Y = 3s - 2.$$

The cubic multiplying Y may be factorised and therefore we have,

$$Y = \frac{3s - 2}{(s^2 + 4)(s + 1)}.$$

Standard partial fractions of the form,

$$\frac{3s - 2}{(s^2 + 4)(s + 1)} = \frac{As + B}{s^2 + 4} + \frac{C}{s + 1},$$

yields $A = 1$, $B = 2$ and $C = -1$. Therefore,

$$\begin{aligned} Y &= \frac{s + 2}{s^2 + 4} - \frac{1}{s + 1}, \\ &= \frac{s}{s^2 + 4} + \frac{2}{s^2 + 4} - \frac{1}{s + 1}. \end{aligned}$$

All three of these terms have standard inverse LTs. Therefore we have,

$$y = \cos 2t + \sin 2t - e^{-t}.$$

3. Find the Laplace Transform of $z(t) = \int_0^t y(\tau) d\tau$. [Hint: recall that $z'(t) = y(t)$ here.]
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We'll integrate by parts once only and differentiate the z -term:

$$\begin{aligned} \mathcal{L}\left[\int_0^t y(\tau) d\tau\right] dt &= \mathcal{L}[z(t)] = \int_0^\infty z(t)e^{-st} dt \\ &= \left[z\right] \left[\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty [z'] \left[\frac{e^{-st}}{-s}\right] dt \\ &= 0 + \frac{1}{s} \int_0^\infty ye^{-st} dt = \frac{Y}{s} \end{aligned}$$

In the above, note that $z(0) = 0$, given its definition.

4. Find the solution of the ODE, $y'' + 2y' + y = 2e^{-t}$, subject to $y(0) = y'(0) = 0$. [Hint: you may need to consult the solution to Q1j.]
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The Laplace Transform of the ODE yields,

$$(s^2 + 2s + 1)Y = \frac{2}{s + 1} \quad \implies \quad Y = \frac{2}{(s + 1)^3}.$$

Question 1j has the solution, $\mathcal{L}[t^2e^{-t}] = 2/(s + 1)^3$, therefore the present solution is $y = t^2e^{-t}$.

5. Factorise the denominator of the following fractions into complex factors, and use partial fractions to find their Inverse Laplace Transforms:
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(a) $\frac{1}{s^2 + b^2} = \frac{1}{(s + bj)(s - bj)} = \frac{1}{2bj} \left[\frac{1}{s - bj} - \frac{1}{s + bj} \right].$

Hence

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2 + b^2}\right] &= \frac{1}{2bj} \mathcal{L}^{-1}\left[\frac{1}{s - bj} - \frac{1}{s + bj}\right] = \frac{1}{2bj} [e^{bjt} - e^{-bjt}] \\ &= \frac{1}{2bj} [(\cos bt + j \sin bt) - (\cos bt - j \sin bt)] = \frac{\sin bt}{b}. \end{aligned}$$

(b) $\frac{s}{s^2 + b^2} = \frac{1}{2} \left[\frac{1}{s - bj} + \frac{1}{s + bj} \right].$

Hence

$$\mathcal{L}^{-1}\left[\frac{s}{s^2 + b^2}\right] = \frac{1}{2} [e^{bjt} + e^{-bjt}] = \cos bt.$$

$$(c) \quad \frac{1}{s^2 + 2cs + c^2 + d^2} = \frac{1}{2dj} \left[\frac{1}{s + c - dj} - \frac{1}{s + c + dj} \right].$$

Note that this exercise is very similar to part (a). The solution is $(1/d)e^{-ct} \sin dt$.

$$(d) \quad \frac{s + c}{s^2 + 2cs + c^2 + d^2} = \frac{1}{2} \left[\frac{1}{s + c - dj} + \frac{1}{s + c + dj} \right]. \text{ The solution is } e^{-ct} \cos dt.$$

Use some of these results to solve the following equations:

$$(e) \quad y'' + 4y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Here we have

$$\mathcal{L}[y''] = s^2Y - sy(0) - y'(0) = s^2Y - 1, \quad \mathcal{L}[y'] = sY - y(0) = sY.$$

The equation transforms to $(s^2 + 4s + 5)Y = 1$ and hence $Y = 1/(s^2 + 4s + 5)$. Now we can use the result of part (c) with $c = 2$ and $d = 1$: $y = e^{-2t} \sin t$.

$$(f) \quad y'' + 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Here we find that $\mathcal{L}[y'] = sY$ and $\mathcal{L}[y''] = s^2Y$. The equation transforms to $(s^2 + 2s + 2)Y = 1/(s + 1)$, from which we obtain

$$Y = \frac{1}{(s + 1)(s^2 + 2s + 2)} = \frac{1}{s + 1} - \frac{s + 1}{s^2 + 2s + 2} \quad \Rightarrow \quad y = (1 - \cos t)e^{-t}$$

after using partial fractions and part (d) with $c = d = 1$.

6. Write down the values of the following integrals.

In all cases we apply the result that the integral of $g(t)\delta(t - a)$ is $g(a)$, unless a lies outside of the range of integration, in which case the result is 0.

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(t)e^{2t} dt &= 1. \\ \int_{-\infty}^{\infty} \delta(t - 1)e^{-t^2} dt &= e^{-1}. \\ \int_{-\infty}^{\infty} \delta(t - 2) \sin \pi t dt &= \sin 2\pi t = 0. \\ \int_0^{\infty} \delta(t + 2)t^3 dt &= 0. \end{aligned}$$

The final answer is zero because the impulse is at $t = -2$ which outside the range of integration. So the impulse function is always zero within that range.

7. Find the Laplace Transforms of the following functions:

(a) $\mathcal{L}[e^{e^t} \delta(t-1)] = \int_0^\infty e^{e^t} \delta(t-1) e^{-st} dt = e^{e^1} e^{-s} = e^{e-s}$ using the result for the integrals of delta functions.

(b) $\mathcal{L}[\delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \dots]$.

As $\mathcal{L}[\delta(t-n)] = e^{-ns}$, this transform is $e^0 + e^{-s} + e^{-2s} + e^{-3s} + \dots$

Another way of writing this is $\sum_{n=0}^\infty e^{-ns}$, which is a geometrical series and may be summed to get $1/(1 - e^{-s})$.

The following summation of unit impulses, $\sum_{n=-\infty}^\infty \delta(t-n)$, (noting the lower limit) is known as the Shah function ($\text{III}(t)$) or, more descriptively, as the Dirac comb. It is used in signal processing and sampling.

8. Use the Laplace Transform to solve the following equations:

(a) $\frac{dy}{dt} + 3y = \delta(t), \quad y(0) = 1.$

The Laplace Transform of the full ODE is

$$(s+3)Y - 1 = 1,$$

and hence,

$$Y = \frac{2}{s+3} \implies y = 2e^{-3t}.$$

Clearly the initial displacement given by the solution is not what was set as the initial condition, but this is due to the impulse.

(b) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \delta(t), \quad y(0) = 1, \quad \frac{dy}{dt}(0) = b$ where b is a constant.

After taking the Laplace Transform the ODE becomes,

$$s^2Y - b - s + 3sY - 3 + 2Y = 1.$$

Hence,

$$(s^2 + 3s + 2)Y = s + 4 + a.$$

After rearrangement and then partial fractions we obtain,

$$Y = \frac{s+4+b}{(s+1)(s+2)} = \frac{3+b}{s+1} - \frac{2+a}{s+2},$$

upon using partial fractions. So the final solution is,

$$y = (3 + b)e^{-t} - (2 + b)e^{-2t}.$$

Although this solution yields $y(0) = 0$, as desired, it is easily shown that $y'(0) = 1 + b$. So a unit momentum has been added.

$$(c) \quad \frac{d^3y}{dt^3} - \frac{dy}{dt} = 3\delta(t), \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0, \quad y''(0) = -1.$$

After the taking of Laplace Transforms we get,

$$s^3Y - sY + 2 - s^2 = 3 \quad \implies \quad Y = \frac{s^2 + 1}{s^3 - s} \quad \implies \quad Y = \frac{1}{s+1} + \frac{1}{s-1} - \frac{1}{s}.$$

Hence the final solution is,

$$y = e^t + e^{-t} - 1.$$

If we check the initial conditions, then we see that $y(0) = 1$ and $y'(0) = 0$, as required. However, $y''(0) = 2$, according to the solution, whereas we imposed $y''(0) = -1$ at the start. We need to bear in mind that the $3\delta(t)$ forcing term for this 3rd order ODE increases the value of $y''(0)$ immediately, and this is what we have seen.

9. Laplace Transforms are perfectly set up to solve Initial Value Problems, but let us try it out on a Boundary Value Problem. The aim, then, is to solve $y'' + y = 0$, subject to $y(0) = 1$ and $y(\frac{1}{2}\pi) = 1$. At the outset, let $y'(0) = c$ and carry out the analysis using this unknown constant. Eventually you will have the opportunity to find c .

Setting $y'(0) = c$, the Laplace Transform of the ODE yields,

$$(s^2 + 1)Y - s - c = 0.$$

Hence,

$$Y = \frac{s + c}{s^2 + 1} = \frac{s}{s^2 + 1} + c \frac{1}{s^2 + 1}.$$

Hence,

$$y = \cos t + c \sin t.$$

Now we are in a position to satisfy the second Boundary Condition, $y(\frac{1}{2}\pi) = 1$; hence $c = 1$. The final solution is,

$$y = \cos t + \sin t.$$

Comment: So clearly it is possible to solve BVPs using Laplace Transforms. I am not sure how tricky this will become if we have to carry more than one unknown initial condition.