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Department of Mechanical Engineering, University of Bath

Mathematics 2 ME10305

Problem Sheet — Fourier Series

Q1. Which of the following functions are even, odd or neither about $x = 0$? Of those which are periodic, find the fundamental period.

(i) $\sin t$ — Odd function. Periodic with period 2π.

(ii) $\sin^2 t$ — Even function. Periodic with period π.

(iii) $\sqrt{1-t^2}$ $(-1 \leq t \leq 1)$ — Even function. Not periodic because it is defined over a finite range. This would still be true even if $f(t) = \cos \pi t$.

(iv) te^{-t} — Asymmetric function. Not periodic.

(v) e^{-t^2} — Even function. Not periodic.

(vi) te^{-t^2} — Odd function. Not periodic.

(vii) $\sin t + \sin 3t$ — Odd function. Periodic with period 2π.

(viii) $\sin t \sin 3t$ — Even function. Periodic with period π. Using the appropriate multiple angle formulae, the function is $\frac{1}{2}(\cos 2t - \cos 4t)$.

(ix) $\sin t \sin \sqrt{2}t$ — Even function. Not periodic. The two sines are periodic, but their periods are incommensurate.

(x) $f(t) = t + 1$ for $-1 < t < 1$, $f(t) = f(t+2)$ — Asymmetric function. Periodic with period 2.

(xi) $f(t) = t$ for $0 \leq t \leq 1$, $f(t) = 2 - t$ for $1 \leq t \leq 2$, $f(t) = f(t+2)$ — Even function. Periodic with period 2.

(xii) $f(t) = 1$ for $0 < t \leq 1$, $f(t) = 2 - t$ for $1 \leq t < 2$, $f(t) = f(t+2)$ — Asymmetric function. Periodic with period 2.

In the last three cases, it would be best to sketch the functions in order to determine their symmetries.

Q2. Find the Fourier Series representations of the following functions, bearing in mind that quicker results may be obtained when symmetries are accounted for. In all cases try to predict in advance how fast the Fourier coefficients decay by checking the continuity of each function **before** attempting to find the Fourier Series.

(a) $f(t) = t^2 \quad -\pi \leq t \leq \pi$ with $f(t) = f(t + 2\pi)$.

(b) $f(t) = t - t^2 \quad 0 \leq t \leq 1$ with $f(t) = f(t+1)$.

(c) $f(t) = \pi^2 t - t^3 \quad -\pi \leq t \leq \pi$ with $f(t) = f(t + 2\pi)$.

(d) $f(t) = t - t^3 \quad -1 \leq t \leq 1$ with $f(t) = f(t+2)$.

(e) $f(t) = \cos \alpha t \quad -1 \leq t \leq 1$ with $f(t) = f(t+2)$.

(f) $f(t) = \cosh \alpha t \quad -1 \leq t \leq 1$ with $f(t) = f(t+2)$.

(g) $f(t) = 1$ for $0 < t < 1$, $f(t) = -1$ for $1 < t < 2$, with $f(t) = f(t+2)$.

(h) $f(t) = 3t^5 - 10t^3 + 7t$ for $-1 \leq t \leq 1$ with $f(t) = f(t+2)$.

(i) $f(t) = |\sin t|$.

(j) $f(t) = t^2$ for $0 < t < 1$ with $f(t) = f(t+1)$.

ANSWER: In a few cases I will fill in the details of the integration by parts. It is important to recall that the Fourier Series consists of sines and cosines of $(2\pi nt/T)$ where T is the period. In each case we apply the standard formula with the correct value of T.

(a) This function is even and therefore B_n, the sine coefficients, are zero. The Fourier coefficients should decay like $1/n^2$ since $f(t)$ is continuous but $f'(t)$ is not. This is because $f(\pi) = f(-\pi)$ and $f'(-\pi) \neq f'(\pi)$. The period is 2π and therefore we have $\sin nt$ and $\cos nt$ terms.

The value of A_0 is given by

$$\begin{aligned}
A_0 &= \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2\,dt && \text{by definition}\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt && \text{cancelling 2s}\\
&= \frac{2}{\pi}\int_{0}^{\pi} t^2\,dt && \text{function is even}\\
&= \frac{2\pi^2}{3}.
\end{aligned}$$

$$\begin{aligned}
A_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2\cos nt\,dt && \text{by definition}\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos nt\,dt && \text{cancelling 2s}\\
&= \frac{2}{\pi}\int_{0}^{\pi} t^2\cos nt\,dt && \text{function is even}\\
&= \frac{2}{\pi}\left[[t^2]\left[\frac{\sin nt}{n}\right] - [2t]\left[\frac{-\cos nt}{n^2}\right] + [2]\left[\frac{-\sin nt}{n^3}\right]\right]_0^{\pi} && \text{by parts}\\
&= \frac{4}{\pi n^2}\Big[t\cos nt\Big]_0^{\pi} && \text{sines=0, simplifying}\\
&= \frac{4}{n^2}(-1)^n. && \cos n\pi = (-1)^n
\end{aligned}$$

Therefore the Fourier series is

$$f(t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nt}{n^2}.$$

(b) Here $T = 1$ and we have $\sin 2\pi nt$ and $\cos 2\pi nt$ terms, in general. However, the function is even, and therefore we will not have sine terms, i.e. $B_n = 0$. Finally, we expect the series to decay like $1/n^2$ because $f(t)$ is continuoous, but $f'(t)$ is not.

$$A_0 = 2\int_0^1 (t - t^2)\,dt = {}^1\!/_3.$$

$$A_n = 2\int_0^1 (t - t^2)\cos 2\pi nt\,dt = -\frac{1}{n^2\pi^2}.$$

Therefore

$$f(t) = \frac{1}{6} - \sum_{n=1}^{\infty}\frac{\cos 2\pi nt}{n^2\pi^2}.$$

(c) This function is odd and therefore its Fourier series consists solely of sines. Its Fourier coefficients should decay like $1/n^3$ because both f and f' are continuous but f'' is not. The period is 2π. Applying the formula we obtain

$$B_n = \frac{2}{2\pi}\int_{-\pi}^{\pi}[\pi^2 t - t^3]\sin nt\,dt = \frac{12(-1)^{n+1}}{n^3}.$$

Hence

$$f(t) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3} \sin nt.$$

Finally, just to note that somewhere in your analysis you'll eventually get $(-\cos n\pi)$ appearing. This is just the same as $-(-1)^n$ or $(-1) \times (-1)^n$ or $(-1)^{n+1}$.

(d) This function is a scaled version of the last one. Again it is odd, but the period is now 2. We get

$$B_n = \frac{2}{2} \int_{-1}^{1} (t - t^3) \sin n\pi t \, dt = \frac{12(-1)^{n+1}}{n^3 \pi^3}.$$

Hence

$$f(t) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3 \pi^3} \sin n\pi t.$$

(e) In this case I have not specified the value of α, but $f(t)$ is always even and continuous, although $f'(t)$ is not; therefore the Fourier coefficients will decay like $1/n^2$.

$$A_0 = \int_{-1}^{1} \cos \alpha t \, dt = 2 \int_{0}^{1} \cos \alpha t \, dt = \frac{2 \sin \alpha}{\alpha}.$$

Note that $A_0 = 2$ when $\alpha = 0$; this may be proved using either L'Hôpital's rule, or by expanding $\sin \alpha$ in a Taylor series about $\alpha = 0$.

$$\begin{aligned}
A_n &= \int_{-1}^{1} \cos \alpha t \, \cos n\pi t \, dt = 2 \int_{0}^{1} \cos \alpha t \, \cos n\pi t \, dt && \text{using symmetry} \\
&= \int_{0}^{1} \Big[\cos(\alpha + n\pi)t + \cos(\alpha - n\pi)t \Big] \, dt && \text{using the appropriate multiple angle formula} \\
&= \Big[\frac{\sin(\alpha + n\pi)}{\alpha + n\pi} + \frac{\sin(\alpha - n\pi)}{\alpha - n\pi} \Big] \\
&= \Big[\frac{\sin \alpha \cos n\pi}{\alpha + n\pi} + \frac{\sin \alpha \cos n\pi}{\alpha - n\pi} \Big] && \text{expanding again with } \sin n\pi = 0 \\
&= \frac{2\alpha}{\alpha^2 - n^2\pi^2} (-1)^n \sin \alpha.
\end{aligned}$$

Hence we obtain

$$f(t) = \frac{\sin \alpha}{\alpha} + \sum_{n=1}^{\infty} \frac{2(-1)^n \alpha \sin \alpha}{\alpha^2 - n^2 \pi^2} \cos n\pi t.$$

I would like to draw your attention to some strange consequences of this Fourier series. When α is not an integer multiple of π, then it is just an ordinary Fourier series. However, when $\alpha = m\pi$, where m is a positive integer, then $\sin \alpha = 0$. This would normally mean that every term is zero, because each term in the series is proportional to $\sin \alpha$. However, when $n = m$ in the summation, we have a zero-divide-zero situation because the denominator is also zero. Again, the use of L'Hôpital's rule as $\alpha \to m\pi$ will yield $A_m = 1$. Hence the Fourier series reduces down to

$$f(t) = \cos m\pi t$$

when $\alpha = m\pi$. Therefore the original function is its own 1-term Fourier series!

The other unusual situation is when $\alpha = 0$. In this case every term in the summation is zero. The constant reduces to precisely 1. Given that $f(t) = \cos 0 = 1$, this means that we again obtain a 1-term Fourier series.

(f) The integrations in this case are a little more difficult than in case (e), because they involve integrations by parts. The result is

$$f(t) = \frac{\sinh \alpha}{\alpha} + \sum_{n=1}^{\infty} \frac{2(-1)^n \alpha \sinh \alpha \cos n\pi t}{\alpha^2 + n^2\pi^2}.$$

The only special case arises when $\alpha = 0$, in which case $f(t) = 1$, and all the terms in the summation disappear. But the degree of similarity between the present Fourier Series and that of Q2e is very strong.

(g) A quick sketch of this function shows that it is odd, and hence all the A coefficients are zero. It has period equal to 2. It is also discontinuous, and therefore its Fourier coefficients will be proportional to $1/n$. We have,

$$\begin{aligned}
B_n &= \frac{2}{2}\int_0^2 f(t)\sin n\pi t\,dt && \text{by definition} \\
&= \int_{-1}^{1} f(t)\sin n\pi t\,dt && \text{limits changed for convenience} \\
& && \text{— see a sketch} \\
&= 2\int_0^1 1\,\sin n\pi t\,dt && \text{using the even symmetry of the integrand} \\
&= 2\Big[-\frac{\cos n\pi t}{n\pi}\Big]_0^1 \\
&= -\frac{2}{n\pi}\Big[\cos n\pi t\Big]_0^1 \\
&= -\frac{2}{n\pi}\Big[\cos n\pi - 1\Big] \\
&= \begin{cases} 4/n\pi & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}$$

Therefore we may write this series in the following way,

$$f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}\sin n\pi t.$$

An alternative way of doing this is the following,

$$f(t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi}\sin(2m+1)\pi t.$$

Here, the factor $(2m+1)$ picks out the odd integers.

(h) $f(t)$ is continuous because $3t^5 - 10t^3 + 7t = 0$ at both $t = -1$ and $t = 1$.
$f'(t)$ is continuous because $15t^4 - 30t^2 + 7 = -8$ at both $t = -1$ and $t = 1$ (and it is even!).
$f''(t)$ is continuous because $60t^3 - 60t = 0$ at both $t = -1$ and $t = 1$.
$f'''(t)$ is continuous because $180t^2 - 60 = 120$ at both $t = -1$ and $t = 1$ (and it is even).
$f''''(t)$ is discontinuous because $360t = \pm 360$ at $t = \pm 1$.

When $f(t)$ is discontinuous then the Fourier coefficients are $\propto 1/n$.
When $f(t)$ is continuous but $f'(t)$ is discontinuous then the Fourier coefficients are $\propto 1/n^2$.
When $f(t)$ and $f'(t)$ are continuous but $f''(t)$ is discontinuous then the Fourier coefficients are $\propto 1/n^3$.
When $f(t)$, $f'(t)$ and $f''(t)$ are continuous but $f'''(t)$ is discontinuous then the Fourier coefficients are $\propto 1/n^4$.
When $f(t)$, $f'(t)$, $f''(t)$ and $f'''(t)$ are continuous but $f''''(t)$ is discontinuous then the Fourier coefficients are $\propto 1/n^5$.

Hence the Fourier coefficients should decay like $1/n^5$ in this case. The detailed computation is:

$$\begin{aligned}
B_n &= \int_{-1}^{1} f(t) \sin n\pi t \, dt && \text{by definition, noting that } f(t) \text{ is odd} \\
&= 2\int_{0}^{1} (3t^5 - 10t^3 + 7t) \sin n\pi t \, dt && \text{using symmetry} \\
&= 2\Bigg[\Big[3t^5 - 10t^3 + 7t\Big]\Big[-\frac{\cos n\pi t}{n\pi}\Big] - \Big[15t^4 - 30t^2 + 7\Big]\Big[-\frac{\sin n\pi t}{n^2\pi^2}\Big] \\
&\qquad + \Big[60t^3 - 60t\Big]\Big[\frac{\cos n\pi t}{n^3\pi^3}\Big] - \Big[180t^2 - 60\Big]\Big[\frac{\sin n\pi t}{n^4\pi^4}\Big] \\
&\qquad + \Big[360t\Big]\Big[\frac{-\cos n\pi t}{n^5\pi^5}\Big] - \Big[360\Big]\Big[\frac{-\sin n\pi t}{n^6\pi^6}\Big] \Bigg]_0^1 \\
&= 2\Big[\frac{360(-\cos n\pi)}{n^5\pi^5}\Big] = \frac{720(-1)^{n+1}}{n^5\pi^5}.
\end{aligned}$$

thereby confirming the above qualitative result. Hence,

$$f(t) = \sum_{n=1}^{\infty} \frac{720(-1)^{n+1}}{n^5\pi^5} \sin n\pi t.$$

(i) This is an interesting function. It has period π, and is even, although it looks at first glance as though it ought to be odd! It is continuous, but its first derivative isn't, and so the coefficients will decay as $1/n^2$.

We will take one period as being from $t = 0$ to $t = \pi$. In this range $f(t) = \sin t$. Hence

$$A_0 = \frac{2}{\pi} \int_0^{\pi} \sin t \, dt = \frac{4}{\pi}.$$

$$\begin{aligned}
A_n &= \frac{2}{\pi} \int_0^{\pi} \sin t \, \cos 2nt \, dt \\
&= \frac{1}{\pi} \int_0^{\pi} \Big[\sin(2n+1)t - \sin(2n-1)t\Big] \, dt && \text{using multiple angle formulae} \\
&= \frac{1}{\pi} \Big[\frac{\cos(2n-1)t}{2n-1} - \frac{\cos(2n+1)t}{2n+1}\Big]_0^{\pi}
\end{aligned}$$

$$= \frac{1}{\pi}\Big[\frac{\cos(2n-1)\pi - 1}{2n-1} - \frac{\cos(2n+1)\pi - 1}{2n+1}\Big]$$

$$= \frac{1}{\pi}\Big[\frac{\cos 2n\pi\cos\pi - 1}{2n-1} - \frac{\cos 2n\pi\cos\pi - 1}{2n+1}\Big] \qquad \text{multiple angle formulae again}$$

$$= \frac{2}{\pi}\Big[\frac{1}{2n+1} - \frac{1}{2n-1}\Big]$$

$$= -\frac{4}{\pi(4n^2-1)}.$$

Therefore the Fourier series is,

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2-1}.$$

(j) This is the only function in this question which is neither even nor odd, and therefore its Fourier Series will have both sines and cosines. The period is 1. The function is discontinuous and therefore the Fourier coefficients will decay as $1/n$. We get,

$$A_0 = 2\int_0^1 t^2\,dt = {}^2\!/_3,$$

$$A_n = 2\int_0^1 t^2\cos 2\pi nt\,dt = \frac{1}{n^2\pi^2},$$

$$B_n = 2\int_0^1 t^2\sin 2\pi nt\,dt = -\frac{1}{n\pi}.$$

Therefore the Fourier series is,

$$f(t) = {}^1\!/_3 + \sum_{n=1}^{\infty}\Big[\frac{\cos 2\pi nt}{n^2\pi^2} - \frac{\sin 2\pi nt}{n\pi}\Big].$$

Although the cosines decay as $1/n^2$, the sines decay as $1/n$, which is slower, and which confirms our original prediction.

Q3. The aim of this question is two-fold, to derive the formulae for the Fourier coefficients and to prove Parseval's theorem. For simplicity we will consider functions of period 2π. **(This question is over and above what would be expected in an exam question, but it is included to show where the formulae for the Fourier coefficients come from.)**

If m and n are nonzero integers, first show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nt\,\cos mt\,dt \quad = \quad \begin{cases} 0 & \text{when } n \neq m, \\ 1 & \text{when } n = m. \end{cases}$$

Do the same for the integral of the product of two sines. Finally, show that the integral of $\sin nt\cos mt$ over the same range is zero.

Hence use these results and the standard definition of the Fourier series,

$$f(t) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos nt + B_n\sin nt),$$

to find expressions for the Fourier coefficients.

For a function of period 2π, Parseval's theorem is

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2dt = \tfrac{1}{2}A_0^2 + \sum_{n=1}^{\infty}[A_n^2 + B_n^2];$$

prove this using the results you have already derived. This result is related to the energy content of a periodic signal.

ANSWER: We may evaluate the integral of the product of cosines by first using a multiple angle formula. We have

$$\cos nt\, \cos mt = \tfrac{1}{2}\big[\cos(n+m)t + \cos(n-m)t\big].$$

Given that we are integrating over a range of length 2π, it is clear that $\cos(n+m)t$ executes $n+m$ full cosine waves in that range, and therefore its integral must be zero. The same will be true for $\cos(n-m)t$, but only when $n \neq m$. When $n = m$, this cosine becomes precisely 1, and therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt\, \cos mt\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} {}^{1}\!/_{2}\, dt = 1.$$

Therefore we have proved the given result.

In similar fashion, we see that

$$\sin nt\, \sin mt = \tfrac{1}{2}\big[-\cos(n+m)t + \cos(n-m)t\big],$$

and precisely the same arguments may be made. Therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt\, \sin mt\, dt \quad = \quad \begin{cases} 0 & \text{when } n \neq m, \\ 1 & \text{when } n = m. \end{cases}$$

For the third integral, namely,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt\, \cos mt\, dt,$$

we get,

$$\sin nt\, \cos mt = \tfrac{1}{2}\big[\sin(n+m)t + \sin(n-m)t\big].$$

The integral from $-\pi$ to $+\pi$ of $\sin(n+m)t$ is also zero, but the corresponding integral of $\sin(n-m)t$ turns out to be zero in all cases; the only possible special case, $n = m$, results in $\sin 0$, which is zero. Therefore all cases yield a zero integral.

We may now derive the Fourier coefficients quite easily. If we start with,

$$f(t) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt), \qquad (*)$$

and integrate it between $-\pi$ and $+\pi$, all the sines and cosines integrate to zero. Therefore,

$$\int_{-\pi}^{\pi} f(t)\, dt = \int_{-\pi}^{\pi} \tfrac{1}{2}A_0\, dt = \pi A_0 \qquad \Rightarrow \qquad A_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt.$$

If we now multiply Eq. (*) by $\cos mt$ and integrate it in like fashion, we get the following,

$$\int_{-\pi}^{\pi} f(t)\cos mt\, dt = \tfrac{1}{2}\int_{-\pi}^{\pi} A_0 \cos mt\, dt + \sum_{n=1}^{\infty}\Big[\int_{-\pi}^{\pi} A_n \cos nt \cos mt\, dt + \int_{-\pi}^{\pi} B_n \sin nt \cos mt\, dt\Big].$$

Using our previous results, the integrals involving A_0 and B_n must be zero. Likewise all the integrals involving A_n, except for the one when the summation index, n, takes the value m. Therefore the above equation reduces immediately to,

$$\int_{-\pi}^{\pi} f(t)\cos mt\, dt = \pi A_m.$$

Hence,

$$A_m = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos mt\, dt.$$

An almost identical argument leads to

$$B_m = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin mt\,dt.$$

Note: the constant outside of these latest integrals have been written in the way they have in order to reflect the standard (2/period) coefficient in the general formula which was presented in the lectures.

Finally, Parseval's theorem may by derived by first squaring both sides of Eq. (*). I'll write it in this form:

$$[f(t)]^2 = \left[\tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos nt + B_n\sin nt)\right]\left[\tfrac{1}{2}A_0 + \sum_{m=1}^{\infty}(A_m\cos mt + B_m\sin mt)\right].$$

Now consider which terms out of the right hand side products will integrate from $-\pi$ to $+\pi$ and yield a nonzero value. It will be those terms for which $n = m$. Therefore,

$$\int_{-\pi}^{\pi}[f(t)]^2\,dt = \int_{-\pi}^{\pi}\Big[\tfrac{1}{4}A_0^2 + \tfrac{1}{2}\Big(A_1^2 + A_2^2 + A_3^2 + \cdots\Big) + \tfrac{1}{2}\Big(B_1^2 + B_2^2 + B_3^2 + \cdots\Big)\Big].$$

All other possible integrands integrate to zero. Hence,

$$\int_{-\pi}^{\pi}[f(t)]^2\,dt = \frac{\pi}{2}A_0^2 + \pi\Big(A_1^2 + A_2^2 + A_3^2 + \cdots\Big) + \pi\Big(B_1^2 + B_2^2 + B_3^2 + \cdots\Big).$$

On dividing by π we obtain the desired result,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt = \tfrac{1}{2}A_0^2 + \sum_{n=1}^{\infty}[A_n^2 + B_n^2].$$

Q4. If $g(t) = t^2$ in the range $-\pi \le t \le \pi$, and $g(t)$ has a period equal to 2π, find its Fourier series. Hence find the Particular Integral of the ordinary differential equation,

$$\frac{dy}{dt} + cy = g(t).$$

ANSWER: A quick sketch is enough to tell us that $g(t)$ is even and therefore all the B_n coefficients are zero. Given that the period is $T = 2\pi$, the general formula for the Fourier coefficients becomes,

$$A_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\,\cos nt\,dt \qquad (n = 0, 1, 2, \cdots,)$$

or, given that $f(t)$ is even, we may use symmetry to obtain the practically simpler formula:

$$A_n = \frac{2}{\pi}\int_{0}^{\pi} f(t)\,\cos nt\,dt \qquad (n = 0, 1, 2, \cdots.)$$

Hence,

$$A_0 = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2\pi^2}{3},$$

and

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} t^2\cos nt\,dt \\ &= \frac{2}{\pi}\Big[\Big(t^2\Big)\Big(\frac{\sin nt}{n}\Big) - \Big(2t\Big)\Big(-\frac{\cos nt}{n^2}\Big) + \Big(2\Big)\Big(-\frac{\sin nt}{n^3}\Big)\Big]_0^{\pi} \\ &= \frac{4\pi\cos n\pi}{\pi n^2} \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

Hence we have,

$$g(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nt.$$

Therefore the equation which will be solved is

$$\frac{dy}{dt} + cy = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nt. \tag{1}$$

The Particular Integral corresponding to the constant term on the Right Hand Side of Eq. (1) is $\pi^2/3c$.

Consider just one of the cosines on the RHS of (1); let

$$\frac{dy}{dt} + cy = A_n \cos nt.$$

By setting $y_{\mathrm{pi}} = C \cos nt + D \sin nt$, we eventually find that $C = A_n c/(c^2 + n^2)$ and $D = A_n n/(c^2 + n^2)$ after a lot of tedious algebra. Therefore the required full solution is,

$$y = \frac{\pi^2}{3c} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2(c^2 + n^2)} \Big[c \cos nt + n \sin nt \Big].$$

Q5. Consider the response of the following undamped/zero resistance system to a rectified sine wave signal:

$$\frac{d^2y}{dt^2} + K^2 y = |\sin t|.$$

By sketching the signal confirm that its period is π and determine its Fourier series. Hence find the response $y(t)$. For which values of K is there resonance?

ANSWER: The term on the RHS of this equation appears in Q2i. It has a period equal to π and is even. The Fourier series is

$$|\sin t| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1}.$$

Therefore we are solving,

$$\frac{d^2y}{dt^2} + K^2 y = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1}.$$

That part of the Particular Integral which corresponds to the constant, $2/\pi$, is $2/(K^2\pi)$.

Given that the Particular Integral of

$$\frac{d^2y}{dt^2} + K^2 y = A \cos 2nt$$

is $[A/(K^2 - 4n^2)] \cos 2nt$, then the solution we seek is,

$$y = \frac{2}{K^2\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(K^2 - 4n^2)(4n^2 - 1)} \cos 2nt.$$

This solution is valid as long as K is not equal to an even number. Therefore resonance happens when K is equal to an even number.

Q6. Find the Fourier series of the response to the following damped system to the rectified sine wave:

$$y'' + cy' + K^2 y = |\sin t|.$$

For which values of K is there resonance?

ANSWER: The algebra required for this problem is quite extensive. The solution is

$$y = \frac{2}{K^2\pi} + \sum_{n=1}^{\infty} \frac{1}{4n^2-1}\left[\frac{(K^2-4n^2)\cos 2nt + 2cn\sin 2nt}{(K^2-4n^2)^2+4c^2n^2}\right].$$

Resonance does not occur when $c \neq 0$, although if K is an even number and c is very small then one of the terms in the series (viz. the one corresponding to $n = \frac{1}{2}K$) will have a large amplitude.
