MATRIX CLASSIFICATION

Square matrices are used ubiquitously in science and engineering, but in the great majority of cases they exhibit some form of structure. The following lists some of the most common structures. Most of these examples use 4×4 matrices, but the classification extends in the obvious way to general $N \times N$ matrices.

General matrix: $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Lower triangular: $$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Upper triangular: $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

These arise when solving systems of algebraic equations.

Diagonal matrix: $$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

Zero matrix: $$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Identity matrix: $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Symmetric: $$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 10 & 101 \\ 3 & 10 & -100 & \pi \\ 4 & 101 & \pi & \sqrt{2} \end{pmatrix}$$

Antisymmetric: $$\begin{pmatrix} 0 & 11 & -2 & 9 \\ -11 & 0 & 31 & 800 \\ 2 & -31 & 0 & 7 \\ -9 & -800 & -7 & 0 \end{pmatrix}$$

Entries in a symmetric matrix satisfy, $a_{ij} = a_{ji}$, but they satisfy $a_{ij} = -a_{ji}$ for an antisymmetric matrix.

Transposition: $$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 9 \\ 1 & 101 & 1 & 1 \end{pmatrix} \quad \Rightarrow \quad A^T = \begin{pmatrix} 1 & 3 & 6 & 1 \\ 2 & 4 & 7 & 101 \\ 3 & 5 & 8 & 1 \\ 4 & 6 & 9 & 1 \end{pmatrix}$$

The **transpose** of a matrix, A, is formed by interchanging its rows and columns. Clearly the transpose of an upper triangular matrix is a lower triangular matrix and vice versa. The transpose of a symmetric matrix is equal to the original matrix ($A = A^T$). The transpose of an antisymmetric matrix satisfies $A^T = -A$. The transpose of a transpose gives the oroginal matrix: $(A^T)^T = A$.

Tridiagonal: $$\begin{pmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 \\ 0 & 0 & 0 & a_5 & b_5 \end{pmatrix}$$

Block tridiagonal: $$\begin{pmatrix} B_1 & C_1 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & 0 & 0 \\ 0 & A_3 & B_3 & C_3 & 0 \\ 0 & 0 & A_4 & B_4 & C_4 \\ 0 & 0 & 0 & A_5 & B_5 \end{pmatrix}$$

Here, a, b and c represent constants, while A, B and C represent $M \times M$ square matrices. The former may arise when solving certain linear ODEs numerically. The latter form arises when solving certain PDEs numerically, and it is almost always the case that the A and C entries are diagonal matrices, while the B-entries are themselves tridiagonal.

Other patterns or types arise such as the periodic tridiagonal matrix, pentadiagonal matrices, Hessenberg matrices, orthogonal matrices, singular matrices, the Hessian matrix, rotation matrices and the Wronskian.

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