Department of Mechanical Engineering, University of Bath

Mathematics 2 ME10305 Sheet 0

The following rather interesting pieces of mathematics were found in this year's Maths 1 examination scripts. The nature of the errors varies substantially from the trivial to the utterly appalling. The following are in the order in which I saw them. Determine what the examinees did incorrectly in each case.

Q1. $\cos 4\theta + \sin 4\theta = (\cos\theta + \sin\theta)^4$

Answer: This should be: $\cos 4\theta + j\sin 4\theta = (\cos\theta + j\sin\theta)^4$

Q2. $y = e^{-\sin t^2} \quad \Rightarrow \quad y' = -2t\sin t^2 e^{-\sin t^2}$

Answer: The sine in the exponent hasn't been differentiated.

Q3. $A = 2\pi \displaystyle\int_a^b (y)^2 \sqrt{1 - \left(\frac{dy}{dx}\right)^2}\, dx$

Answer: The formula for the surface area of revolution was derived using Pythagoras's theorem on a small element of area. This means that the minus ought to be a plus.

Q4. $t^{-2}e^{2t} \quad \Rightarrow \quad \dfrac{d}{dt} = e^{2t}(2t^{-2} - 2t^{-3})$

Answer: The left hand side of the proffered answer is a differential operator while the right hand side is a function of t. These are not the same things. The differential operator is waiting for a function of t to differentiate while the right hand side is a function of t; one should not use this notation to mean, 'hence the derivative is'! Either one should set $y = t^{-2}e^{2t}$ and then write $\dfrac{dy}{dt} = e^{2t}(2t^{-2} - 2t^{-3})$, or else one may write the answer as

$$\frac{d(t^{-2}e^{2t})}{dt} = e^{2t}(2t^{-2} - 2t^{-3}).$$

Q5. $\displaystyle\int_0^\pi t^2 \cos t\, dt = \cdots = -2\pi = 2\pi$

Answer: My guess is that the students, for there were many, expected a positive answer but couldn't find the error in their analysis. Therefore the sign was changed. The integral is negative, as a quick sketch will suggest.

Q6. $\displaystyle\int \frac{dt}{2t} = \ln|2t+2| + c$

Answer: The error becomes clear if one replaces the original integral by $\frac{1}{2}\displaystyle\int \frac{dt}{t} = \frac{1}{2}\ln|t| + c$. This is the same kind of mistake as one would have with,

$$\frac{d\sin 2t}{dt} = \cos 2t.$$

Q7. Given $y = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n}$, then $|x| < \displaystyle\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 2$. Hence the radius of convergence is 2.

Answer: This formula, where one uses a_n, is not one which I have taught even though one will find it in every textbook. It is used when the series takes the form, $\sum_{n=0}^{\infty} a_n x^n$ [i.e. $u_n = a_n x^n$ The formula which I have taught applies to the more general case, $\sum_{n=0}^{\infty} u_n$. Both ways work well if the series involving the a_n values are precisely as given above. In the exam question we don't have x^n but we do have x^{2n}. The application of d'Alembert's test where $u_n = x^{2n}/2^n$ will give $x^2 < 2$ for convergence, which is correct. Although I was also taught the way with a_n, I far far prefer to use what I taught you, for it will cope with all the different type of power series, whereas the a_n–formula doesn't. I take a unified approach where I don't need to worry cases and when the maths is done all is well.

Q8. $V = \int_0^1 \int_0^2 (x+2y)^2 \, dx \, dy = -4/3$

Answer: Bing bing bing.....it's an integral of a square and therefore the answer MUST be positive. You must always have this sort of secondary checking mechanism fizzing away in your brain every time you do some maths. Clearly there is a mistake.

Q9. $\int_0^1 x^3 \sqrt{1+9x^4} \, dx = \int_0^1 x^{3/2} + 9x^5 \, dx$

Answer: This is a version of $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, well, almost, for the '9' remains intact and the x^3 has been raised to the half power. Also note the absence of brackets... see the answer to Q24.

Q10. $\int_0^1 x^3 \sqrt{1+9x^4} \, dx = \int_0^1 x^3(1+3x^2) \, dx$

Answer: Almost the same....

Q11. $(x, y) = (0, 3)$ is a straddle point

Answer: A saddle point, even if one can straddle a saddle.

Q12. $t = 1$ is a minima

Answer: The word *minima* is plural. One should say that $t = 1$ is a minimum.

Q13. $|\underline{\mathrm{b}}| = \sqrt{6} = 2.45$

Answer: There's no real need to approximate $\sqrt{6}$ when it is exact. But if one does wish to approximate it, then three significant figures is quite inaccurate; use five or six.

Q14. $\int_0^{2\pi} (x \sin x)^2 \, dx = \int_0^{2\pi} (x^2 + 2x \sin x + \sin^2 x) \, dx$

Answer: Oops, the x and the $\sin x$ are being multiplied, not added.

Q15. $\int \frac{1}{2t} \, dt = \frac{1}{2} \ln |t| = \ln |t^{1/2}|$

Answer: The middle term here is the correct answer. The last one is fine only if it guaranteed that $t > 0$. This solution, $\ln |t|^{1/2}$, is also correct.

Q16. $\begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$

Answer: A vector can never be equal to scalar even when they are both zero.

Q17. $\lim_{x \to 0} \left(\frac{1 - \cos x}{3x^2} \right) = \lim_{x \to 0} \left(\frac{1 + \sin x}{6x} \right) = \infty$

Answer: A surprisingly common error from the exam paper. The derivative of 1 is 0, not 1.

Q18. $\dfrac{t+2}{t^3-t} = \dfrac{t+2}{t^2(t-1)} = \dfrac{A}{t} + \dfrac{B}{t^2} + \dfrac{C}{t-1}$

Answer: Another surprisingly common one. Incorrect factorisation of the denominator.

Q19. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx = \tfrac{1}{4}\pi + c$

Answer: Definite integrals do not have arbitrary constants.

Q20. $u_n = \dfrac{(-1)^n x^{2n}}{2^n} \Rightarrow u_{n+1} = \dfrac{(-1)^n x^{2n+1}}{2^n}$

Answer: When the value n becomes $n+1$, all the instances of n in the formula for u_n must be replaced by $n+1$. Thus $2n$ becomes $2(n+1)$, or $2n+2$.

Q21. $\displaystyle\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1^n)x^n$

Answer: This must be $(-1)^n$. The taking of powers takes precedence over subtraction.

Q22. $\displaystyle y' = \sum_{n=0}^{\infty}(-x)^n \Rightarrow y = c + \sum_{n=0}^{\infty}\frac{(-x)^{n+1}}{n+1}$

Answer: It is safer to think first that $(-x)^n$ is equal to $(-1)^n x^n$. Then its integral is $(-1)^n x^{n+1}/(n+1)$.

Q23. $\displaystyle\int_0^{2\pi} x\sin x\,dx = \Big[(-\cos x)(x) - (-\sin x)(1)\Big]_0^{2\pi} = -6.136$

Answer: This is perhaps the trickiest one of the lot. The correct answer should be -2π which is -6.2832 to four decimal places. Being so close to the correct answer actually gave me the clue. The student had his/her calculator computing in degrees not radians. Usually this would give something which is very different for the correct answer, but here the value of the sine is very small but not zero. Frankly, this should not have been computed using the calculator because we all know the sine and cosine of zero and 2π.

Q24. Exam: $\displaystyle V = \int_0^1 \int_0^2 (x+2y)^2\,dx\,dy$

Script: $\displaystyle V = \int_0^1 \int_0^2 x + 2y^2\,dx\,dy$

Answer: There are some people (even academics in this university, given what I have seen on the UniHall whiteboards!) who write

$$\int t^2 + t^3\,dt \quad \text{instead of} \quad \int (t^2+t^3)\,dt.$$

The limiting definition of the integral makes it clear that one is adding up the areas of all of those little strips: Area $= \sum(t^2+t^3)\delta t$, and therefore the brackets are the proper way of writing the integral of a sum. The above example from a script has further horrors....