UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2021	
Unit Title: Mathematics 2		Unit Code: ME10305	
Year: 2020/21	Question Number: 1	Page 1 of 1	

Part		Mark
(a)	Let $y_1 = f$, $y_2 = g$, $y_3 = g'$, $y_4 = h$, $y_5 = h'$, $y_6 = h''$ $\Rightarrow$ $y_1' = y_2$, $y_2' = y_3$, $y_3' = y_4 - y_1 y_2$, $y_4' = y_5$, $y_5' = y_6$, $y_6' = y_2 - e^{-x} y_1 - \frac{1}{2} y_1 y_6$. $x=0$: $y_1 = 1$, $y_4 = 0$, $y_5 = 1$, $y_6 = -5$. $x=1$: $y_6 = 0$	3
(b)	6^{th} order, nonlinear BVP	2
(c)	$\frac{dy}{dx} = y\cos 2x \Rightarrow \int \frac{dy}{y} = \int \cos 2x\, dx \Rightarrow \ln y = \frac{\sin 2x}{2} + c$ $\Rightarrow y = Ae^{\frac{\sin 2x}{2}}$. IC: $y(0) = 2 \Rightarrow A = 2$ $\Rightarrow \boxed{y = 2e^{\frac{\sin 2x}{2}}}$	2
(d)	$xy' + (1+x)y = (1+\cos x)e^{-x} \longrightarrow y' + \frac{(1+x)}{x} y = \frac{1+\cos x}{x} e^{-x}$ Int. Factor is $e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = xe^{x}$. Eq $\rightarrow$ $xe^{x}y' + (1+x)e^{x}y = 1 + \cos x$ $\Rightarrow xe^{x}y = x + \sin x + c$ $\Rightarrow y = e^{-x} + \frac{\sin x}{x}e^{-x} + \frac{c}{x}e^{-x}$. $y(\pi) = 2 \Rightarrow 2 = e^{-\pi} + \frac{c}{\pi}e^{-\pi}$ $\Rightarrow c = \pi(2e^{\pi} - 1)$. Hence $y = e^{-x} + \frac{\sin x}{x}e^{-x} + \frac{\pi(2e^{\pi}-1)e^{-x}}{x}$ $\Rightarrow \boxed{y = e^{-x}\left[1 + \frac{\sin x}{x} + \frac{\pi}{x}(2e^{\pi} - 1)\right]}$.	3
	Total	10

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Part		Mark
(a)	$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$ $\quad \mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$ $\quad \mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$ $\mathcal{L}[f(t)e^{-at}] = F(s+a)$	2
(b)	$s^3+s-10 = (s-2)(s^2+2s+5)$	2
(c)	Let $\frac{3s+1}{s^3+s-10} = \frac{A}{s-2} + \frac{Bs+C}{s^2+2s+5}$ $= \frac{7/13}{s-2} + \frac{\left(-\frac{7}{13}s + \frac{11}{13}\right)}{s^2+2s+5}$ after partial fractions $= \frac{7/13}{s-2} - \frac{\frac{7}{13}(s+1)}{s^2+2s+5} + \frac{\frac{9}{13}\times 2}{(s+1)^2+2^2}$ Hence $\mathcal{L}^{-1}\left[\frac{3s+1}{s^3+s-10}\right] = \frac{7}{13}e^{2t} - \frac{7}{13}e^{-t}\cos 2t + \frac{9}{13}e^{-t}\sin 2t$ [n.b. a lot has been missed out! This includes the use of the above shift theorem]	6
		Total 10

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Part		Mark
(a)	$\begin{vmatrix} 1 & 3 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 2 & 2 & 1 & -1 \\ 2 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 \end{vmatrix} = +2 \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 2 & 2 & 1 \end{vmatrix}$ $(+C_3)$ $= 2 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 2 & -1 \end{vmatrix} = 2\times(-1)\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 2\times(-1)\times(-2)$ $(-C_1)$ $= \boxed{4}$	4
(b)	Augmented matrix notation. $\left(\begin{array}{ccc\|c} 1 & 4 & 1 & 4 \\ 3 & 16 & 5 & 14 \\ -1 & 4 & 4 & -1 \end{array}\right)$ $\left(\begin{array}{ccc\|c} 1 & 4 & 1 & 4 \\ 0 & 4 & 2 & 2 \\ 0 & 8 & 5 & 3 \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + R_1 \end{array}$ $\left(\begin{array}{ccc\|c} 1 & 4 & 1 & 4 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$	4
	$\Rightarrow z = -1$ $\Rightarrow y = 1$ $\Rightarrow x = 1$	2
	Full marks for Gaussian elimination Half - - - - . Gauss–Jordan Zero - - - . anything else. G.E. was asked for.	

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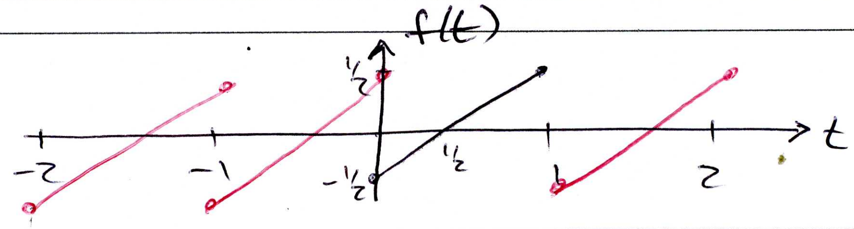
Part		Mark
(a)		2
(b)	Period $=1$. Function is odd $\Rightarrow$ need only the B_n terms. So $A_0=0$, $A_n=0$ by symmetry. $B_n = 2\int_0^1 (t-\tfrac{1}{2})\sin 2n\pi t\,dt$ $= 2\Big[(t-\tfrac12)\big(-\frac{\cos 2n\pi t}{2n\pi}\big) - (1)\big(-\cancel{\frac{\sin 2n\pi t}{4n^2\pi^2}}\big)\Big]_0^1$ $= 2\Big[\tfrac12\big(-\frac{1}{2n\pi}\big) - \big(-\tfrac12\big)\big(-\frac{1}{2n\pi}\big)\Big] = -\frac{1}{n\pi}$ Hence $\boxed{f(t)=\sum_{n=1}^{\infty}\left(-\frac{1}{n\pi}\right)\sin 2n\pi t}$	4
(c)	$y''+4y=f(t)$ $\Rightarrow \boxed{y=\sum_{n=1}^{\infty}\frac{\sin 2n\pi t}{n\pi(n^2\pi^2-4)}}$	3
(d)	1^{st} derivative is continuous, but not the second.	1
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Part		Mark
(a)	$r_i = y_i - a x_i^2 - b x_i \implies S = \sum_{i=1}^{N} (y_i - a x_i^2 - b x_i)^2$	
	Need $0 = \frac{\partial S}{\partial a} = \sum -2x_i^2 (y_i - a x_i^2 - b x_i)$	
	& $0 = \frac{\partial S}{\partial b} = \sum -2 x_i (\ldots\ldots)$	
	$\implies \begin{pmatrix} \sum x_i^4 & \sum x_i^3 \\ \sum x_i^3 & \sum x_i^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i x_i^2 \\ \sum y_i x_i \end{pmatrix}$	4
	Evaluating the sums:	
(b)	$\begin{pmatrix} 1.5664 & 1.8 \\ 1.8 & 2.2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1.92808 \\ 2.2396 \end{pmatrix}$	
	$\implies \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1.02143 \\ 0.18229 \end{pmatrix}$ 5DPs	6

Total	10

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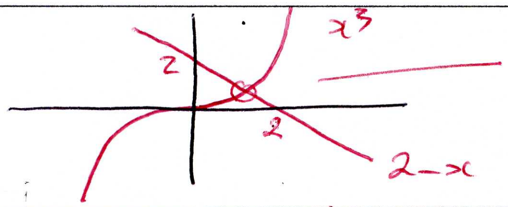
Part		Mark
(a)	 1 root	2
(b)	A: $x_{n+1} = 2 - x_n^3$ Diverges $1.1 \longrightarrow 0.669 \longrightarrow 1.700582 \longrightarrow -2.918015 \longrightarrow 26.847114$ B: $x_{n+1} = (2-x_n)^{1/3}$ Converges $1.1 \longrightarrow 0.965489 \longrightarrow 1.011374 \longrightarrow 0.996194 \longrightarrow 1.001267$	2
(c)	A: $x_{n+1} = 2-(1+\varepsilon)^3 = 2-(1+3\varepsilon \cdots) = 1-3\varepsilon \cdots$ Divergence B: $x_{n+1} = (2-(1+\varepsilon))^{1/3} = (1-\varepsilon)^{1/3} = 1-\frac{1}{3}\varepsilon \cdots$ Convergence	2
(d)	NR: $x_{n+1} = x_n - \left(\dfrac{x_n^3+x_n-2}{3x_n^2+1}\right) = \dfrac{2x_n^3+2}{3x_n^2+1}$ $1.2 \longrightarrow 1.025564 \longrightarrow 1.000488 \longrightarrow 1.000000$	2
(e)	$x_{n+1} = \dfrac{2(1+\varepsilon)^3+2}{3(1+\varepsilon)^2+1} = \dfrac{4+6\varepsilon+6\varepsilon^2+2\varepsilon^3}{4+6\varepsilon+3\varepsilon^2}$ $= 1 + \dfrac{3\varepsilon^2+2\varepsilon^3}{4+6\varepsilon+3\varepsilon^2}$ $= 1 + \frac{3}{4}\varepsilon^2 + \cdots$ Quadratic convergence	2

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Part		Mark
(a)	$0 = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)(4-\lambda)-1\right] - (3-\lambda) + \left[1-(2-\lambda)\right]$ $= (2-\lambda)(\lambda^2-6\lambda+7) - (3-\lambda) + (\lambda-1)$ $= -\lambda^3+8\lambda^2-19\lambda+14 - 4+2\lambda$ $= -\lambda^3+8\lambda^2-17\lambda+10 \qquad [\lambda=1 \text{ is a root}]$ $= (\lambda-1)(-\lambda^2+7\lambda-10) = -(\lambda-1)(\lambda-2)(\lambda-5).$ Eigenvalues are $\lambda = 1, 2, 5$ $\lambda=1: \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+y+z=0 \\ x+y+3z=0 \end{cases} \Rightarrow z=0, \; x+y=0$ Use $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\lambda=2: \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\lambda=5: \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$	6
(b)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + Be^{2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + Ce^{5t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	2
(c)	IC: $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + C\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ By inspection $C=1, B=-1, A=0$ $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{5t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - e^{2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$	2

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Part		Mark
	Take LT of the two ODEs:	

$$s^2X-1+5X-4Y=0, \quad s^2Y-1-4X+5Y=0$$

$$\Rightarrow \begin{pmatrix} s^2+5 & -4 \\ -4 & s^2+5 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{(s^2+5)^2-16}\begin{pmatrix} s^2+5 & 4 \\ 4 & s^2+5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{s^4+10s^2+9}\begin{pmatrix} s^2+9 \\ s^2+9 \end{pmatrix}$$

$$= \frac{1}{(s^2+1)(s^2+9)}\begin{pmatrix} s^2+9 \\ s^2+9 \end{pmatrix}$$

$$= \begin{pmatrix} 1/s^2+1 \\ 1/s^2+1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin t \\ \sin t \end{pmatrix}$$

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Part		Mark
(a)	Using $y \propto e^{\lambda t} \Rightarrow \lambda^3 + 3\lambda^2 + \lambda + 3 = 0$ $\Rightarrow (\lambda^2+1)(\lambda+3) = 0$ $\Rightarrow y_{CF} = Ae^{-3t} + B\cos t + C\sin t$ $y_{PI} = 1.$ ICs $y(0)=1 \Rightarrow 1 = A + B + 1$ $y'(0)=1 \Rightarrow 1 = -3A + C$ $y''(0)=2 \Rightarrow 2 = 9A - B$ $A = \frac{1}{5}$, $B = -1/5$, $C = 8/5$ $\Rightarrow \boxed{y = \frac{1}{5}e^{-3t} - \frac{1}{5}\cos t + \frac{8}{5}\sin t + 1}$	5
(b)	$\underbrace{y'' + 4y}_{\lambda = \pm 2j} = \underbrace{4\cos 2t}_{\lambda = \pm 2j}$ $\Rightarrow y_{CF} = A\cos 2t + B\sin 2t$ Let $y_{PI} = Ct\cos 2t + Dt\sin 2t$ $\Rightarrow y''_{PI} = C[-4t\cos 2t - 4\sin 2t] + D[-4t\sin 2t + 4\cos 2t]$ $\Rightarrow y''_{PI} + 4y_{PI} = -4C\sin 2t + 4D\cos 2t = 4\cos 2t$ $\Rightarrow D = 1,\ C = 0$ $\Rightarrow$ Gen sol: $y = A\cos 2t + B\sin 2t + t\sin 2t$ IC $y(0)=0 \Rightarrow A = 0$ $y'(0) = 2 \Rightarrow 2 = 2B \Rightarrow B = 1$ $\Rightarrow \boxed{y = (1+t)\sin 2t}$	5

	Total
	10

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Part		Mark

(a) Let $I = \int_{-\pi}^{\pi} r^2\,dx = \int_{-\pi}^{\pi} [f - a - b\cos x]^2\,dx$

$$= \int_{-\pi}^{\pi} \Big[f^2 + a^2 + b^2 \underbrace{\cos^2 x}_{\frac12 + \frac12\cos 2x} - 2af - 2bf\cos x + 2ab\cos x\Big]\,dx$$

$$= \int_{-\pi}^{\pi} f^2\,dx + 2\pi a^2 + \pi b^2 - 2a\int_{-\pi}^{\pi} f - 2b\int_{-\pi}^{\pi} f\cos x + 0$$

So $0 = \dfrac{\partial I}{\partial a} = 4\pi a - 2\int_{-\pi}^{\pi} f\,dx \Rightarrow \boxed{a = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$

& $0 = \dfrac{\partial I}{\partial b} = 2\pi b - 2\int_{-\pi}^{\pi} f\cos x\,dx \Rightarrow \boxed{b = \dfrac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx}$

5

(b) These are Fourier coefficients ($a = \frac12 A_0$, $b = A_1$)

2

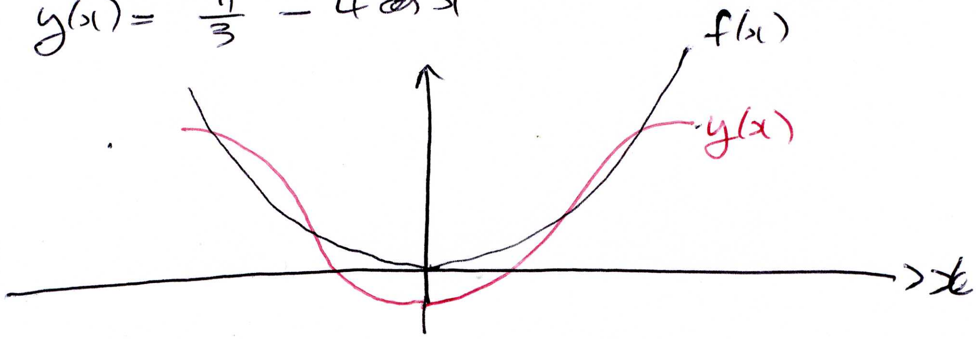
(c) $a = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \dfrac{1}{\pi}\int_{0}^{\pi} x^2\,dx = \dfrac{\pi^2}{3}$

$b = \dfrac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos x\,dx = \dfrac{2}{\pi}\int_{0}^{\pi} x^2\cos x\,dx$

$= \dfrac{2}{\pi}\Big[(x^2)(\cancel{\sin x}) - (2x)(-\cos x) + (2)(-\cancel{\sin x})\Big]_0^{\pi}$

$= -4$

$\Rightarrow y(x) = \dfrac{\pi^2}{3} - 4\cos x$



3

Total	10