Name: DASR

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Q1a. The equations become: [2 marks]

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = \tfrac{1}{2}y_2^2 - \tfrac{3}{4}y_1y_3 - y_4$$
$$y_4' = y_5$$
$$y_5' = -\tfrac{3}{4}y_1y_5$$

The boundary conditions become: [1 mark]

$x=0$: $y_1=0,\ y_2=0,\ y_4=1$

$x\to\infty$: $y_2\to 0,\ y_4\to 0$

Q1b. Classification: 5th order, nonlinear BVP [2 marks]

Q1c. Solution: $y = +(2e^{t^2}-1)^{1/2}$ [2 marks]

Q1d. Integrating Factor: $\frac{1}{t}e^{-t}$ [1 mark]

Solution: $y = e^t(1-t)$ [2 marks]

Q2a. Solutions: $\frac{1}{s+a}$ $\quad 2/s^3$ [2 marks]

Q2b. Solution: $\frac{1}{2}t^2e^{-t}$ [4 marks]

Q2c. Solution: $3/5\,e^{-3t} + \frac{2}{5}e^{2t}$ [4 marks]

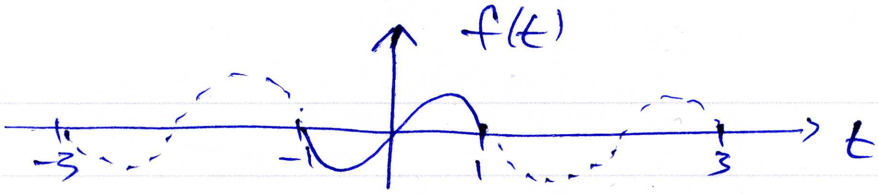
Q3a. Determinant: -4 [4 marks]

Q3b. Upper triangular form: [4 marks]

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 16 \end{pmatrix}$$

Final solution: $\begin{pmatrix} 8 \\ 27 \\ 8 \end{pmatrix}$ [2 marks]

Q4a. Sketch: [2 marks]



Q4b. Fourier Series: $\displaystyle\sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3} \sin n\pi t$ [4 marks]

Q4c. Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3(4-n^2\pi^2)} \sin n\pi t$ [3 marks]

Q4d. Answer: 4th derivative is discontinuous at $t=1$ [1 mark]

3 are continuous

Q5a. Formula being used: $\begin{pmatrix} \sum x_i^4 & \sum x_i^3 \\ \sum x_i^3 & \sum x_i^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i x_i^2 \\ \sum y_i x_i \end{pmatrix}$ [3 marks]

Q5b. Numerical solution: $\begin{pmatrix} -0.9871 \\ 3.0987 \end{pmatrix}$ [7 marks]

Q6a. Factorisation: $(x-1)(x-3)(x+4)$ [1 mark]

Q6b. Ad hoc schemes: A: $x_{n+1} = (13x_n - 12)^{1/3}$ [1 mark]

B: $x_{n+1} = (x_n^3 + 12)/13$

Iterations: A: $1.1 \to 1.320006 \to 1.728033 \to 2.187284 \to 2.542549$ [2 marks]

B: $1.1 \to 1.025461 \to 1.006027 \to 1.001399 \to 1.000323$

Q6c. $x_{n+1} =$ $(1+13\varepsilon)^{1/3} \simeq 1 + \frac{13}{3}\varepsilon$ [1 mark]

$x_{n+1} =$ $(1 + 3/13\,\varepsilon)$ [1 mark]

Q6d. Iteration scheme: $x_{n+1} = \dfrac{2x_n^3 - 12}{3x_n^2 - 13}$ [1 mark]

Iterates: $1.2 \to 0.984332 \to 0.999978 \to 1.000000$ [1 mark]

Q6e. $x_{n+1} =$ $1 - \frac{3}{10}\varepsilon^2$ [2 marks]

Q7a. Eigenvalues and eigenvectors: [6 marks]

$\lambda=-1$: $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\quad\lambda=3$: $\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad\lambda=-3$: $\begin{pmatrix}1\\-2\\1\end{pmatrix}$

Q7b. General solution: [3 marks]

$$\underline{x} = Ae^{-t}\begin{pmatrix}1\\0\\-1\end{pmatrix} + Be^{3t}\begin{pmatrix}1\\1\\1\end{pmatrix} + Ce^{-3t}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

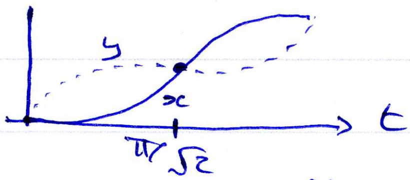
Q7c. Final solution: [1 mark]

$$\underline{x} = e^{3t}\begin{pmatrix}1\\1\\1\end{pmatrix} + e^{-3t}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

Q8a. Solutions: $x = \frac{1}{2}t - \frac{1}{2\sqrt{2}}\sin\sqrt{2}t$ [7 marks]

$y = \frac{1}{2}t + \frac{1}{2\sqrt{2}}\sin\sqrt{2}t$

Q8b. Sketch: [1 mark]



Reason: $y(0)$ violates I.C. because the unit impulse adds a unit momentum. [2 marks]

Q9a. Solution: [5 marks]

$$y = e^{-t}\cos t + t - 1$$

Q9b. Solution: $y = \frac{1}{13}\left[-5e^{-3t} + 5\cos 2t + 12\sin 2t\right]$ [5 marks]

Q10a. Integrating Factor: r^2 [2 marks]

Solution: $y = \frac{1}{r^2} + 2\ln r - 1$ [3 marks]

Q10b. Equation for y in terms of x: $y' + 2y = 4x$ [2 marks]

Solution in terms of x: $y = Ae^{-2x} + 2x - 1$ [2 marks]

Solution in terms of r: $y = \frac{A}{r^2} + 2\ln r - 1 \quad (A=1).$ [1 mark]