UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2020
Unit Title: Mathematics 2		Unit Code: ME10305
Year: 2019/20	Question Number: 1	Page 1 of 1

Part		Mark
(a)	Let $\left.\begin{array}{l} y_1 = f \\ y_2 = f' \\ y_3 = f'' \\ y_4 = g \\ y_5 = g' \end{array}\right\} \Rightarrow \begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = \frac{1}{2}y_2^2 - \frac{3}{4}y_1y_3 - y_4 \\ y_4' = y_5 \\ y_5' = -\frac{3}{4}y_1y_5 \end{array}$ $\quad x=0 \Rightarrow y_1=0, y_2=0, y_4=1$; $\quad x\to\infty \Rightarrow y_2 \to 0, y_4 \to 0$	(2)
	5th order, nonlinear BVP	(2)
(b)	$y\frac{dy}{dt} = t(1+y^2) \Rightarrow \frac{y}{1+y^2}\,dy = t\,dt \Rightarrow \ln(1+y^2) = t^2+c$ $\Rightarrow 1+y^2 = Ae^{t^2}$ or $y^2 = Ae^{t^2}-1$. IC $\Rightarrow y(0)=1 \Rightarrow A=2 \Rightarrow y = +\sqrt{2e^{t^2}-1}$	(3)
(c)	$\div t \Rightarrow \frac{dy}{dt} - (1+\frac{1}{t})y = -\frac{e^{+t}}{t}$. Integrating factor is $e^{-\int(1+\frac{1}{t})dt} = e^{-t-\ln t} = \frac{1}{t}e^{-t}$ Eq $\longrightarrow \frac{1}{t}e^{-t}y' - (\frac{1}{t}+\frac{1}{t^2})e^{-t}y = -\frac{1}{t^2}$ Integrate $\longrightarrow \frac{1}{t}e^{-t}y = 1/t + c$ $\Rightarrow y = e^t + cte^t$ But $y(1)=0 \Rightarrow 0 = e + ce \Rightarrow c=-1$ $\Rightarrow y = e^t(1-t)$.	(3)

Total	10

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(a)	$\mathcal{L}[e^{-at}] = \int_0^\infty e^{-at} e^{-st}\,dt = 1/_{s+a}$	(1)
	$\mathcal{L}[t^2] = \int_0^\infty t^2 e^{-st}\,dt = 2/_{s^3}$ — int. by parts.	(1)
(b)	$\dfrac{1}{s^3+3s^2+3s+1} = \dfrac{1}{(s+1)^3} = F(s+1)$	
	where $F(s) = 1/_{s^3} \Rightarrow f(t) = \frac{1}{2}t^2$.	
	Hence $\mathcal{L}^{-1}\left[\dfrac{1}{(s+1)^3}\right] = \frac{1}{2}t^2 e^{-t}$ using s-shift.	(4)
(c)	$\dfrac{s}{s^2+s-6} = \dfrac{s}{(s+3)(s-2)} = \dfrac{3/5}{s+3} + \dfrac{2/5}{s-2}$	(4)
	$\Rightarrow \mathcal{L}^{-1}\left[\dfrac{s}{s^2+s-6}\right] = 3/_5\, e^{-3t} + 2/_5\, e^{2t}$	
	Total	10

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Part		Mark
(a)	$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & -1 \\ 2 & 2 & 4 & -2 \\ 1 & 2 & 2 & -1 \end{vmatrix} \begin{array}{l} R_1 \\ R_2-R_1 \\ R_3-2R_2 \\ R_4-R_1 \end{array} \longrightarrow \begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 3 & -2 \\ 0 & -2 & 0 \\ 0 & 2 & -2 \end{vmatrix}$ Other routes available. $= -4.$	4
(b)	$\left(\begin{array}{ccc\|c} 0 & 1 & -3 & 3 \\ 1 & -1 & 2 & -3 \\ 3 & -2 & 5 & 10 \end{array}\right)$ Need to swap rows: $\left(\begin{array}{ccc\|c} 1 & -1 & 2 & -3 \\ 0 & 1 & -3 & 3 \\ 3 & -2 & 5 & 10 \end{array}\right) \begin{array}{l} \\ R_2 \\ R_3-R_1 \end{array}$ $\left(\begin{array}{ccc\|c} 1 & -1 & 2 & -3 \\ 0 & 1 & -3 & 3 \\ 0 & 1 & -1 & 19 \end{array}\right) \begin{array}{l} \\ \\ R_3-R_2 \end{array}$ Other routes available (e.g. swapping rows 1&3 at the start) $\left(\begin{array}{ccc\|c} 0 & -1 & 2 & -3 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 2 & 16 \end{array}\right)$ Back substitution $\Rightarrow z = 8$ $y = 27$ $x = 8.$	6
	Total	10

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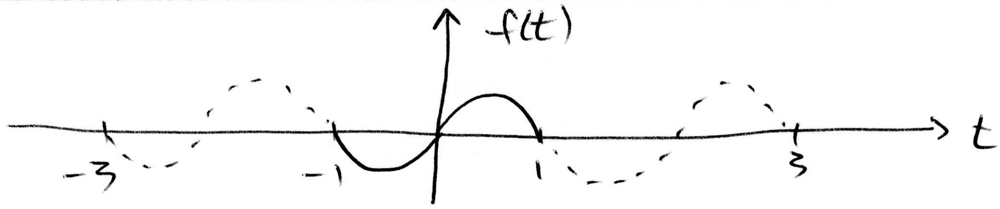
Part		Mark
(a)		2
	Odd function $\Rightarrow A_n = 0$	1
	$B_n = \int_{-1}^{1}(t-t^3)\sin n\pi t\,dt = 2\int_0^1 (t-t^3)\sin n\pi t\,dt$	
	$= 2\Big[(t-t^3)\Big(-\frac{\cos n\pi t}{n\pi}\Big) - (1-3t^2)\Big(-\frac{\sin n\pi t}{n^2\pi^2}\Big) + (-6t)\Big(\frac{\cos n\pi t}{n^3\pi^3}\Big) - (-6)\Big(\frac{\sin n\pi t}{n^4\pi^4}\Big)\Big]_0^1$	
	$= -\frac{12}{n^3\pi^3}\cos n\pi = \frac{12(-1)^{n+1}}{n^3\pi^3}$	
	$\Rightarrow f(t) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3}\sin n\pi t$	3
(b)	Solution of ODE is	
	$y = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3(4-n^2\pi^2)}\sin n\pi t$	3
	Coefficients decay as n^{-5} 3 derivatives are continuous.	1
	Total	10

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Part		Mark
.	Derivation. Fitting $y = ax^2 + bx$.	
	$S = \sum_{i=1}^{N} (y_i - ax_i^2 - bx_i)^2$	
	$\frac{\partial S}{\partial a} = -2\sum_{i=1}^{N} x_i^2 (y_i - ax_i^2 - bx_i) = 0$	
	$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{N} x_i (\cdots\cdots) = 0$	
	$\Rightarrow \begin{pmatrix} \Sigma x_i^4 & \Sigma x_i^3 \\ \Sigma x_i^3 & \Sigma x_i^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \Sigma y_i x_i^2 \\ \Sigma y_i x_i \end{pmatrix}$	4
	Summations yield	
	$\begin{pmatrix} 354 & 100 \\ 100 & 30 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -39.5747 \\ -5.7520 \end{pmatrix}$	
	$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{620} \begin{pmatrix} 30 & -100 \\ -100 & 354 \end{pmatrix} \begin{pmatrix} -39.5747 \\ -5.7520 \end{pmatrix}$	
	$= \begin{pmatrix} -0.98771 \\ 3.0987 \end{pmatrix}$	
	$\therefore y = 3.0987x - 0.98771x^2$.	6
	[used $y = \pi x - x^2 + \text{noise}$].	
		Total 10

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Part		Mark
(a)	$x=1$ is a root $\Rightarrow x^3-13x+12 = (x-1)(x^2+x-12) = (x-1)(x-3)(x+4)$	2
(b)	A: $x_{n+1} = (13x_n-12)^{1/3}$ $1.1 \to 1.320006 \to 1.728033 \to 2.187284 \to 2.542549$ B: $x_{n+1} = (x_n^3+12)/13$ $1.1 \to 1.025461 \to 1.006027 \to 1.001399 \to 1.000323$	2
(c)	A: $x_{n+1} = (13+13\varepsilon-12)^{1/3} = (1+13\varepsilon)^{1/3} \simeq 1+\frac{13}{3}\varepsilon$ diverges B: $x_{n+1} = (1+3\varepsilon+\dots+12)/13 = 1+\frac{3}{13}\varepsilon\dots$ converges	2
(d)	$x_{n+1} = x_n - \dfrac{x_n^3-13x_n+12}{3x_n^2-13} = \dfrac{2x_n^3-12}{3x_n^2-13}$ $1.2 \to 0.984337 \to 0.999928 \to 1.000000$	2
(e)	$x_{n+1} = \dfrac{2(1+3\varepsilon+3\varepsilon^2\dots)-12}{3(1+2\varepsilon+\varepsilon^2)-13}$ $= \dfrac{-10+6\varepsilon+6\varepsilon^2\dots}{-10+6\varepsilon+3\varepsilon^2\dots} = 1 + \dfrac{3\varepsilon^2}{-10+\dots}$ $\simeq 1 - \frac{3}{10}\varepsilon^2.$	2
	Total	10

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Part		Mark

(a)

$$0 = \begin{vmatrix} -\lambda & 2 & 1 \\ 2 & -1-\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = -\lambda\left[\lambda^2+\lambda-4\right] - 2\left[-2\lambda-2\right] + \left[4+1+\lambda\right]$$

$$= -\lambda^3-\lambda^2+4\lambda+4\lambda+4+5+\lambda = -\lambda^3-\lambda^2+9\lambda+9$$

$$= (\lambda+1)(\lambda^2-9) \implies \lambda = -1, 3, -3$$

$\underline{\lambda=-1}$ $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ If $x=1 \Rightarrow z=-1$ (row 2) Row 1 $\Rightarrow y=0$ $\implies \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\underline{\lambda=3}$ $\begin{pmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ If $x=1$ in row 1 $\Rightarrow 2y+z=3$
" " " 2 $\Rightarrow -2y+z=-1$
$\Rightarrow y=z=1$ $\implies \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda=-3}$ $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ If $x=1$ in row 1 $\Rightarrow 2y+z=-3$
" " " 2 $\Rightarrow 2y+2z=-2$
$\Rightarrow z=1, y=-2$ $\implies \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (7)

(b)

$$\underline{x} = Ae^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + Ce^{-3t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$ (2)

$t=0 \Rightarrow \underline{x} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$. $\Rightarrow A=0$
$B=1$
$C=-1$

$$\therefore \underline{x} = e^{3t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - e^{-3t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$ (1)

Total	10

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(a) LT of the ODEs give

$$(s^2+1)X - Y = 0 \qquad (s^2+1)Y - X = 1$$

Eliminate Y to get $[(s^2+1)^2 - 1]X = 1$

$$\Rightarrow X = \frac{1}{s^4+2s^2} = \frac{1}{s^2(s^2+2)} = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+2}\right]$$

Using the given results we have

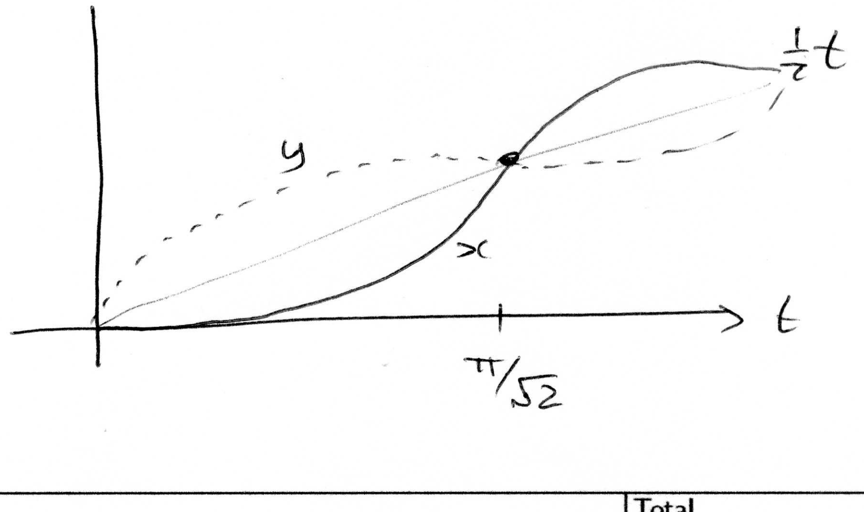
$$x = \frac{1}{2}t - \frac{1}{2\sqrt{2}}\sin\sqrt{2}t.$$

But $y = \frac{d^2x}{dt^2} + x \Rightarrow y = \frac{1}{2}t + \frac{1}{2\sqrt{2}}\sin\sqrt{2}t$ 7

$$\dot{x}(0) = \frac{1}{2} - \frac{\sqrt{2}}{2\sqrt{2}} = 0$$

$$\dot{y}(0) = \frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{2}} = 1$$

The $\dot{y}(0)$ value violates the given IC because the unit impulse acting on the right hand mass imparts a unit momentum.



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(a)	$y'' + 2y' + 2y = 2t.$	
	CF: $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow (\lambda+1)^2 + 1 = 0 \Rightarrow \lambda = -1 \pm j.$	
	$\Rightarrow y_{CF} = e^{-t}[A\cos t + B\sin t]$	
	PI. $y_{PI} = Ct + D \Rightarrow C = 1$ & $D = -1$	
	$\Rightarrow \underline{y_{PI} = t - 1.}$ Hence $y = e^{-t}[A\cos t + B\sin t] + t - 1$	
	I.C. $y(0) = 0 \Rightarrow A = 1.$	
	$y'(0) = 0 \Rightarrow 0 = e^{-t}[-A\cos t - B\cancel{\sin} t - A\cancel{\sin} t + B\cos t] + 1 \Big\vert_{t=0}$	
	$= B - A + 1 \Rightarrow B = 0$	5
	Hence $\boxed{y = e^{-t}\cos t + t - 1}$	
(b)	$y'' + 6y' + 9y = 13\cos 2t$	
	CF $(\lambda^2 + 6\lambda + 9) = 0 \Rightarrow (\lambda+3)^2 = 0 \Rightarrow \lambda = -3, -3$	
	$\Rightarrow y_{CF} = (A + Bt)e^{-3t}.$	
	P.I.: $y_{PI} = C\cos 2t + D\sin 2t$ or $y_{PI} = \mathrm{Re}[Ce^{2jt}].$	
	Eventually: $y_{PI} = \dfrac{5\cos 2t + 12\sin 2t}{13}$	
	$\Rightarrow y = (A + Bt)e^{-3t} + \dfrac{5\cos 2t + 12\sin 2t}{13}$	
	$y(0) = 0 \Rightarrow A + 5/13 = 0 \Rightarrow A = -5/13$	
	$y'(0) = 3 \Rightarrow 3 = \left[e^{-3t}(B - 3A - 3Bt) + \dfrac{-10\sin 2t + 24\cos 2t}{13}\right]_{t=0}$	
	$= B - 3A + \frac{24}{13}$	
	$\Rightarrow B = 3 + 3A - \frac{24}{13} = 3 - \frac{15}{13} - \frac{24}{13} = 0.$	
	Hence $\boxed{y = \frac{1}{13}[-5e^{-3t} + 5\cos 2t + 12\sin 2t]}$	5
	Total	10

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(a)	$\div r \Rightarrow \frac{dy}{dr} + \frac{2}{r}y = \frac{4}{r}\ln r$	
	I.F. is $e^{\int \frac{2}{r}dr} = e^{2\ln r} = r^2 \Rightarrow r^2\frac{dy}{dr} + 2ry = 4r\ln r$	
	$\Rightarrow (r^2y)' = 4r\ln r \Longrightarrow r^2y = c + 4\int r\ln r\,dr$	
	$\left[\int r\ln r\,dr = \ldots \frac{r^2}{2}\ln r - \frac{r^2}{4}\right] \Rightarrow r^2y = c + 2r^2\ln r - r^2$	
	$\Rightarrow y = \frac{c}{r^2} + 2\ln r - 1$	
	I.C. is $y(1)=0 \Rightarrow c=1 \Rightarrow \boxed{y = \frac{1}{r^2} + 2\ln r - 1}$	5
(b)	Let $r = e^x \Rightarrow \frac{dy}{dx} = \frac{dy}{dr}\frac{dr}{dx} = e^x\frac{dy}{dr} = r\frac{dy}{dr}$	
	ODE $\longrightarrow \frac{dy}{dx} + 2y = 4x.$	
	CF is $y_{CF} = Ae^{-2x}$	
	$y_{PI} = 2x - 1$	
	$\Rightarrow y = Ae^{-2x} + 2x - 1$	
	$= A/r^2 + 2\ln r - 1$	
	I.C. as above $\Rightarrow \boxed{y = \frac{1}{r^2} + 2\ln r - 1}$	5
	Total	10