University of Bath

## DEPARTMENT OF MECHANICAL ENGINEERING <br> ME10305 <br> MATHEMATICS 2

23rd May 2023
09:30-11:30 (2 Hours)

## Each question carries 10 marks.

The examination consists of TEN questions. All questions should be attempted.
The marks shown against each part of a question are for guidance only.

Only calculators provided by the University may be used.
During this examination you are not permitted to communicate with any person except for an invigilator or an assigned support worker.

You must not have any unauthorised devices or materials with you.
You must keep your Library card on your desk at all times.

Please fill in the details on the front of your answer book/cover. Sign in the section on the right of your answer book/cover, peel away the adhesive strip and seal.

Take care to enter the correct candidate number as detailed on your desk label. Do not turn over your question paper until instructed to by the chief invigilator.

## Question 1.

(a) Write the following ordinary differential equation and its boundary conditions in first order form:

$$
\frac{d^{4} y}{d x^{4}}+y \frac{d y}{d x}+2 y=1
$$

where the boundary conditions are that,

$$
y=\frac{d y}{d x}=0 \quad \text { at both } \quad x=0 \quad \text { and } \quad x=1 .
$$

Classify this equation and its boundary conditions with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem (giving reasons).
(b) Use the method of Integrating Factors to solve the equation,

$$
\frac{d y}{d t}+\frac{y}{t+1}=1
$$

subject to the initial condition, $y(0)=1 / 2$.
(c) Use the technique of separation of variables to solve the equation,

$$
\left(t^{2}+1\right) \frac{d v}{d t}=-4 v t
$$

subject to $v=1$ when $t=0$.

## Question 2.

The Laplace Transform of $f(t)$ is defined according to

$$
F(s)=\mathcal{L}[f(t)]=\int_{0}^{\infty} f(t) e^{-s t} d t
$$

(a) Use the above definition to find the Laplace Transform of the function $\cos \omega t$.
(b) Use the above definition of the Laplace Transform to find the transform of $f^{\prime \prime}(t)$ in terms of $F(s), f(0)$ and $f^{\prime}(0)$.
(c) Hence solve the equation,

$$
y^{\prime \prime}+y=3 \cos 2 t
$$

subject to the initial conditions, $y(0)=0$ and $y^{\prime}(0)=0$.

## Question 3.

(a) Find the determinant of the matrix,

$$
A=\left(\begin{array}{cccc}
1 & 3 & -1 & 1 \\
2 & 1 & -1 & 1 \\
1 & 2 & -1 & 1 \\
2 & 0 & 1 & 3
\end{array}\right)
$$

(b) Use Gaussian Elimination to solve the following system of simultaneous equations:

$$
\left(\begin{array}{ccc}
-1 & 2 & 1 \\
-2 & 5 & 3 \\
1 & -3 & 0
\end{array}\right)\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)=\left(\begin{array}{c}
2 \\
7 \\
-3
\end{array}\right)
$$

[Note: Full marks will only be awarded for a correct answer which has used the Gaussian Elimination method in its precise form, i.e. where the matrix has been reduced to upper triangular form.]

## Question 4.

(a) Use the definition of the Newton-Raphson iteration scheme for $f(x)=x^{3}-8=0$, to show that the formula given below may be used to find the cube root of 8 :

$$
x_{n+1}=\frac{2 x_{n}^{3}+8}{3 x_{n}^{2}} .
$$

Verify that this formula does find the cube root of 8 by evaluating four successive iterates beginning with $x_{1}=3$.
(b) Analyse the approach to the limit $x_{\infty}=2$ by setting $x_{n}=2+\epsilon_{n}$ in the above formula, where $\epsilon_{n}$ is very small, and by evaluating $x_{n+1}$.

## Question 5.

(a) The function, $f(t)$, has a period equal to $2 \pi$ and it is given by $f(t)=t$ within the range, $-\pi<t<\pi$. Sketch the function in the range $-3 \pi<t<3 \pi$ and find its Fourier series representation. (The definitions of the Fourier coefficients are given below.)
(b) Hence find the particular integral of the ordinary differential equation,

$$
\frac{d^{2} y}{d t^{2}}+\pi^{2} y=f(t)
$$

Clearly, $y(t)$ will be continuous at $t=\pi$, but how many of its derivatives will also be continuous at that point?

You may use the following definitions of the Fourier Coefficients,

$$
f(x)=\frac{1}{2} A_{0}+\sum_{n=1}^{\infty}\left[A_{n} \cos n t+B_{n} \sin n t\right]
$$

where

$$
\begin{array}{ll}
A_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos n t d t \quad n=0,1, \cdots, \infty \\
B_{n}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin n t d t \quad n=1,2, \cdots, \infty
\end{array}
$$

## Question 6.

The following data have been obtained from an experiment.

$$
\begin{array}{c|ccccc}
x_{i} & 1 & 1.25 & 1.5 & 1.75 & 2 \\
y_{i} & 1.01 & 1.22 & 1.46 & 1.63 & 1.83
\end{array}
$$

It is proposed to fit the curve, $y=a+b \sqrt{x}$, to this data using the method of least squares. First develop the theory, and then find the curve of best fit.

## Question 7.

The matrix, $A$, is defined as follows:

$$
A=\left(\begin{array}{cc}
10 & 4 \\
9 & 10
\end{array}\right)
$$

(a) Find all the eigenvalues and eigenvectors of $A$.
(b) Using the results of part (a), determine the general solution of the following system of ordinary differential equations,

$$
\frac{d}{d t}\binom{x}{y}=\left(\begin{array}{cc}
10 & 4 \\
9 & 10
\end{array}\right)\binom{x}{y} .
$$

(c) Hence find that solution which satisfies the initial conditions,

$$
x(0)=1, \quad y(0)=0 .
$$

(d) Using the information found in parts (a) and (b), find the general solution of the following system of ordinary differential equations,

$$
\frac{d^{2}}{d t^{2}}\binom{x}{y}=\left(\begin{array}{cc}
10 & 4 \\
9 & 10
\end{array}\right)\binom{x}{y}
$$

## Question 8.

Solve the ordinary differential equation,

$$
\frac{d^{2} y}{d t^{2}}+y=2 \cos t
$$

subject to the initial conditions

$$
y=0, \quad \frac{d y}{d t}=1 \quad \text { at } \quad t=0 .
$$

## Question 9.

The Laplace Transform of $f(t)$ is defined in Question 2.
(a) Use the definition of the Laplace Transform to find both $\mathcal{L}\left[e^{-a t}\right]$ and $\mathcal{L}[t]$.
(b) Use the definition of the Laplace Transform to prove the $s$-shift theorem

$$
\mathcal{L}\left[f(t) e^{-a t}\right]=F(s+a)
$$

where $\mathcal{L}[f(t)]=F(s)$.
(c) Use the $s$-shift theorem to determine the inverse Laplace Transform of

$$
\frac{1}{s^{2}+4 s+4}
$$

(d) The convolution theorem is $\mathcal{L}[f * g]=F(s) G(s)$, where

$$
\mathcal{L}[f(t)]=F(s), \quad \mathcal{L}[g(t)]=G(s) \quad \text { and } \quad f * g=\int_{0}^{t} f(\tau) g(t-\tau) d \tau
$$

Use the convolution theorem as an alternative way to find the inverse Laplace Transform of

$$
\frac{1}{s^{2}+4 s+4}
$$

## Question 10.

Use the substitution $y(t)=z(t) e^{-2 t}$ to transform the ordinary differential equation,

$$
\frac{d^{2} y}{d t^{2}}+4 \frac{d y}{d t}+4 y=2 e^{-2 t}
$$

into an ordinary differential equation for $z(t)$. Solve this equation for $z$ and hence write down the general solution for $y(t)$. What is the specific solution which satisfies the initial conditions, $y(0)=1$ and $y^{\prime}(0)=0$ ?

