UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees	Date May 2023
Unit Title: Mathematics 2	Unit Code: ME10305
Year: 2022/23 Question Number: 1	Page 1 of 1

Part		Mark
(a)	Let $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $y_4 = y'''$ $\Rightarrow$ $y_1' = y_2$, $y_2' = y_3$, $y_3' = y_4$, $y_4' = 1 - 2y_1 - y_1 y_2$ BCs $y_1 = y_2$ @ both $x = 0, 1$.	2
	4^{th} order, nonlinear BVP	2
(b)	$y' + \frac{1}{t+1} y = 1$ I.F. is $e^{\int \frac{dt}{t+1}} = e^{\ln(t+1)} = (t+1)$ $\Rightarrow (t+1)y' + y = t+1$ $\Rightarrow (t+1)y = \frac{t^2}{2} + t + c$ $y(0) = \frac{1}{2} \Rightarrow c = \frac{1}{2}$ $\Rightarrow (t+1)y = \frac{1}{2}(t^2 + 2t + 1) = \frac{(t+1)^2}{2}$ $\Rightarrow \boxed{y = \frac{1}{2}(t+1)}$	3
(c)	$\frac{dv}{v} = \frac{-4t}{t^2+1} dt$ $\Rightarrow \ln v = -2\ln(t^2+1) + c$ $\Rightarrow v = A(t^2+1)^{-2}$ $v(0) = 1 \Rightarrow A = 1 \Rightarrow \boxed{v = \frac{1}{(t^2+1)^2}}$	3
	Total	10

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(a) Use integration by parts or

$$\mathcal{L}[\cos\omega t] = \mathrm{Re}\,\mathcal{L}[e^{j\omega t}] = \mathrm{Re}\int_0^\infty e^{j\omega t} e^{-st} .$$

$$= \mathrm{Re}\,\frac{1}{s-j\omega} = \mathrm{Re}\,\frac{s+j\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$$ 3

(b) $\mathcal{L}[f'']$ . . . bookwork . . . $s^2F(s) - f'(0) - sf(0)$ 3

(c) $y'' + y = 3\cos 2t \qquad y(0) = 0, \; y'(0) = 0$

$$\Rightarrow \quad s^2 Y + Y = \frac{3s}{s^2+4}$$

$$\Rightarrow \quad Y = \frac{3s}{(s^2+1)(s^2+4)}$$

$$= \frac{s}{s^2+1} - \frac{s}{s^2+4}$$ using partial fractions

$$\Rightarrow y = \cos t - \cos 2t$$ 4

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(a)	$\begin{vmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 3 \end{vmatrix} = \underset{\substack{\uparrow \\ c_3+c_4}}{\begin{vmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 3 \end{vmatrix}} = -4\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$ $= -4\begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} R_1-R_3 = -(-4)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 4$	4
(b)	$\left(\begin{array}{ccc\|c} -1 & 2 & 1 & 2 \\ -2 & 5 & 3 & 7 \\ 1 & -3 & 0 & -3 \end{array}\right)$ $\left(\begin{array}{ccc\|c} -1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 1 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$ $\left(\begin{array}{ccc\|c} -1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right) \begin{array}{l} \\ \\ R_3+R_2 \end{array}$ } Half marks if not upper-triangular Back subst $\Rightarrow z=1,\ y=2,\ x=3$ $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$	6
	Total	10

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Part		Mark
(a)	$x_{n+1} = x_n - f(x_n)/f'(x_n)$ $\Rightarrow x_{n+1} = x_n - \frac{(x_n^3-8)}{3x_n^2} = \frac{2x_n^3+8}{3x_n^2}$.	2
	Using this: 3 → 2.296296 → 2.036587 → 2.000653 → 2.0000002	3
(b)	$x_n = 2+\varepsilon \Rightarrow x_{n+1} = \frac{2(8+12\varepsilon+6\varepsilon^2+\varepsilon^3)+8}{3(4+4\varepsilon+\varepsilon^2)}$ $= \frac{24+24\varepsilon+12\varepsilon^2+2\varepsilon^3}{12+12\varepsilon+3\varepsilon^2}$ $= \frac{2(12+12\varepsilon+3\varepsilon^2)+6\varepsilon^2+2\varepsilon^3}{12+12\varepsilon+3\varepsilon^2}$ $= 2 + \frac{\varepsilon^2}{2} + \dots$ Quadratic convergence.	5
	Total	10

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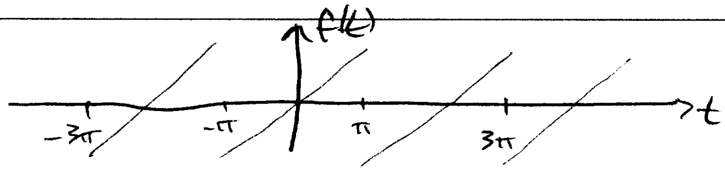
Part | Mark

(a)



2

Odd $\Rightarrow A_n = 0$

$$B_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t \sin nt \, dt = \frac{2}{\pi}\int_0^{\pi} t \sin nt \, dt$$

(symmetry)

$$= \frac{2(-1)^{n+1}}{\pi n}$$

$$\Rightarrow \boxed{f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nt}$$

4

(b)

$$y'' + \pi^2 y = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nt$$

$$\Rightarrow \boxed{y = \sum_{n=1}^{n} \frac{2(-1)^{n+1}}{n(\pi^2 - n^2)} \sin nt}$$

3

One derivative is continuous at $t = \pi$: $y'(\pi)$.

1

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Part | Mark

Let $S = \sum_i (y_i - a - b\sqrt{x_i})^2$

Need $0 = \frac{\partial S}{\partial a} = -2\sum_i (y_i - a - b\sqrt{x_i}) \Rightarrow \boxed{aN + b\sum x_i = \sum y_i}$

$0 = \frac{\partial S}{\partial b} = -2\sum_i \sqrt{x_i}(y_i - a - b\sqrt{x_i}) \Rightarrow \boxed{a\sum\sqrt{x_i} + b\sum x_i = \sum y_i\sqrt{x_i}}$

$$\Rightarrow \underline{\begin{pmatrix} N & \sum\sqrt{x_i} \\ \sum\sqrt{x_i} & \sum x_i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum \sqrt{x_i}\, y_i \end{pmatrix}}$$ 5

Hence $\begin{pmatrix} 5 & 6.079868 \\ 6.079868 & 7.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 7.15 \\ 8.906427 \end{pmatrix}$ 3

$\Rightarrow a = -0.980748$

$b = 1.982569$ 2

$$\boxed{y \approx -0.980748 + 1.982569\sqrt{x}}$$

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Part		Mark
(a)	$0 = \begin{vmatrix} 10-\lambda & 4 \\ 9 & 10-\lambda \end{vmatrix} = (\lambda-10)^2-36 \Rightarrow \lambda = 10 \pm 6 = 4, 16.$ $\underline{\underline{\lambda=4}} \Rightarrow \begin{pmatrix} 6 & 4 \\ 9 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ $\underline{\underline{\lambda=16}} \Rightarrow \begin{pmatrix} -6 & 4 \\ 9 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = B\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	4
(b)	If $\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t}\begin{pmatrix} X \\ Y \end{pmatrix}$ then $\begin{pmatrix} 10-\lambda & 4 \\ 9 & 10-\lambda \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Hence $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 2 \\ -3 \end{pmatrix}e^{4t} + B\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{16t}$	2
(c)	$x(0)=1,\ y(0)=0 \Rightarrow A = B = 1/4$ $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ -3 \end{pmatrix}e^{4t} + \frac{1}{4}\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{16t}$	2
(d)	Same as (b) except that $\lambda^2 = 4, 16 \Rightarrow \lambda = 2, -2, 4, -4$ Solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \left[Ae^{2t} + Be^{-2t}\right]\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \left[Ce^{4t} + De^{-4t}\right]\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	2
	Total	10

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Part		Mark
	$\underline{y'' + y = 2\cos t}$	
	CF: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm j \Rightarrow \boxed{y_{CF} = A\cos t + B\sin t}$	3
	PI: $\lambda = \pm j$ again	
	Let $y_{PI} = Ct\cos t + Dt\sin t$	
	Subst into ODE to get	
	$-2C\sin t + 2D\cos t = 2\cos t$	
	$\Rightarrow C = 0, D = 1.$	
	$\Rightarrow \underline{y_{PI} = t\sin t}$	5
	$y_{gs} = A\cos t + B\sin t + t\sin t$	
	$y(0) = 0 \Rightarrow A = 0$	
	$y'(0) = 1 \Rightarrow B = 1$	
	$\Rightarrow y = (t+1)\sin t$	2
	Total	10

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Part		Mark
(a)	$\mathcal{L}[e^{-at}] = \int_0^\infty e^{-(s+a)t}\,dt = \frac{1}{s+a}$	1
	$\mathcal{L}[t] = \int_0^\infty t e^{-st}\,dt = \ldots = 1/s^2$	1
(b)	$\mathcal{L}[f(t)e^{-at}] = \int_0^\infty f(t)e^{-(s+a)t}\,dt = F(s+a)$	2
(c)	$F(s) = 1/s^2 \Rightarrow F(s+2) = \frac{1}{(s+2)^2} = \frac{1}{s^2+4s+4}$ So $f(t)=t \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^2+4s+4}\right] = te^{-2t}$	3
(d)	As $\frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$, then let $F=G=\frac{1}{s+2}$ Hence $f(t)=g(t)=e^{-2t}$ from part (a). Convolution theorem $\Rightarrow$ $\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t} * e^{-2t}$ $= \int_0^t e^{-2\tau} e^{-2(t-\tau)}\,d\tau$ $= e^{-2t}\int_0^t 1\,d\tau = te^{-2t}$	3
	Total	10

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$y = ze^{-2t} \Rightarrow y' = (z'-2z)e^{-2t} \Rightarrow y'' = (z''-4z'+4z)e^{-2t}$

Hence $y''+4y'+4y = 2e^{-2t}$ becomes

$$\left[(z''-\cancel{4z'}+\cancel{4z}) + 4(\cancel{z'}-\cancel{2z}) + \cancel{4z}\right]e^{-2t} = 2e^{-2t}$$

$\Rightarrow z'' = 2$ 4

$\Rightarrow z = t^2 + At + B$

$\Rightarrow y = [t^2 + At + B]e^{-2t}$ 1

ICs $y(0)=1, \Rightarrow B = 1$

$y'(0)=0$. But $y'(t) = -2[t^2+At+B]e^{-2t} + [2t+A]e^{-2t}$

$\Rightarrow 0 = -2B + A \quad \Rightarrow A = 2.$

Hence $y = (t^2+2t+1)e^{-2t}$

$= (t+1)^2 e^{-2t}$ 5

Total 10