University of Bath

DEPARTMENT OF MECHANICAL ENGINEERING

ME10305

MATHEMATICS 2

23rd May 2022
09:30 – 11:30 (2 Hours)

Each question carries 10 marks.

The examination consists of TEN questions. All questions should be attempted.

The marks shown against each part of a question are for guidance only.

Only calculators provided by the University may be used.

During this examination you are not permitted to communicate with any person except for an invigilator or an assigned support worker.

You must not have any unauthorised devices or materials with you.

You must keep your Library card on your desk at all times.

Please fill in the details on the front of your answer book/cover. Sign in the section on the right of your answer book/cover, peel away the adhesive strip and seal.

Take care to enter the correct candidate number as detailed on your desk label.

Do not turn over your question paper until instructed to by the chief invigilator.

Question 1.

(a) Write the following system of ordinary differential equations and its boundary conditions in first order form:

$$\frac{d^3 f}{dx^3} + f\frac{d^2 f}{dx^2} - g = 0, \qquad \frac{d^2 g}{dx^2} + f\frac{dg}{dx} = 0,$$

subject to,

$$f = \frac{df}{dx} = 0 \quad \text{and} \quad g = 1 \quad \text{at} \quad x = 0,$$

and

$$\frac{df}{dx} \to 1, \quad g \to 0 \quad \text{as} \quad x \to \infty.$$

[2 marks]

(b) Classify the system given in part (a) with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem (giving reasons). [2 marks]

(c) Use the technique of separation of variables to solve the equation,

$$(t^2 + 1)\frac{dy}{dt} = 2yt,$$

subject to $y(0) = 1$. [3 marks]

(d) Find the solution of the equation,

$$\frac{dy}{dt} - \frac{3y}{t} = \frac{4}{t^2},$$

subject to $y = 0$ when $t = 1$, using the integrating factor method. [3 marks]

Question 2.

The Laplace Transform of $f(t)$ is defined according to

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}\,dt.$$

(a) Use the above definition of the Laplace Transform to find the Laplace Transform of the function $\sin \omega t$. [3 marks]

(b) Use the above definition of the Laplace Transform to find $\mathcal{L}[f''(t)]$ in terms of $F(s)$, $f(0)$ and $f'(0)$. [2 marks]

(c) Hence solve the equation,

$$y'' + 9y = 5\sin 2t,$$

subject to the initial conditions, $y(0) = 0$ and $y'(0) = 5$. [5 marks]

Question 3.

(a) Find the determinant of the following matrix:

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

[3 marks]

(b) Use the method of Gaussian Elimination to solve the following system of simultaneous equations:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

[7 marks]

[Note: Full marks will only be awarded for a correct answer which has used the Gaussian Elimination method in its precise form, i.e. where the matrix has been reduced to upper triangular form.]

Question 4.

The following data have been obtained from an experiment.

x_i	0	1	2	3	4
y_i	0.02	1.25	4.02	9.21	15.75

It is proposed to fit the quadratic curve, $y = ax^2 + b$, to this data using the method of least squares. Develop the theory, and then find the line of best fit. [7 marks]

Suppose now that the fitted curve must pass through the origin. What is now the curve of best fit? [3 marks]

Question 5.

(a) The function, $f(t)$, has a period equal to 2 and it is given by $f(t) = 1 - t^2$ within the range, $-1 < t < 1$. Sketch the function in the range $-3 < t < 3$ and find its Fourier series representation. (The definitions of the Fourier coefficients are given below.) [6 marks]

(b) Hence find the particular integral of the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 4y = f(t).$$

Clearly, $y(t)$ will be continuous at $t = 1$, but how many of its derivatives will also be continuous at that point? [4 marks]

You may use the following definitions of the Fourier Coefficients,

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \cos n\pi t + B_n \sin n\pi t\Big],$$

where

$$A_n = \int_{-1}^{1} f(t)\,\cos n\pi t\,dt \qquad n = 0, 1, \cdots, \infty,$$

$$B_n = \int_{-1}^{1} f(t)\,\sin n\pi t\,dt \qquad n = 1, 2, \cdots, \infty.$$

Question 6.

(a) The aim for this question is to employ some iteration schemes to solve the equation, $x^2 - 5x + 4 = 0$. First solve the equation analytically. [1 mark]

(b) There are two possible *ad hoc* schemes for solving this quadratic equation numerically. Write one of them down, and use this formula to evaluate the first three iterates beginning with $x_0 = 1.1$. [2 marks]

(c) Analyse the rate of convergence of this *ad hoc* scheme by setting $x_n = 1 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating x_{n+1}. [2 marks]

(d) Write down the Newton-Raphson scheme for this quadratic. Use this formula to evaluate the first two iterates beginning again with $x_0 = 1.1$. [2 marks]

(e) Analyse the approach to the limit $x_\infty = 1$ for this Newton-Raphson scheme by setting $x_n = 1 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating x_{n+1}. [3 marks]

Question 7.

The matrix, A, is defined as follows:

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -3 \end{pmatrix}.$$

(a) Find all the eigenvalues and eigenvectors of A. [6 marks]

(b) Hence write down the general solution of the system of ordinary differential equations,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

[2 marks]

(c) What is the solution of the equations given in part (b) if the initial condition is that $(x, y, z) = (2, 0, 2)$ at $t = 0$? [2 marks]

Question 8.

The Laplace Transform of $f(t)$ is defined according to,

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt.$$

(a) Use the definition of the Laplace Transform to find both $\mathcal{L}[e^{-at}]$ and $\mathcal{L}[\,t\,]$. [2 marks]

(b) Use the definition of the Laplace Transform to prove the s-shift theorem

$$\mathcal{L}[f(t)e^{-at}] = F(s+a)$$

where $\mathcal{L}[f(t)] = F(s)$. [2 marks]

(c) Use the s-shift theorem to determine the inverse Laplace Transform of

$$\frac{1}{s^2 + 10s + 25}.$$

[3 marks]

(d) The convolution theorem is $\mathcal{L}[f * g] = F(s)G(s)$, where

$$\mathcal{L}[f(t)] = F(s), \quad \mathcal{L}[g(t)] = G(s) \quad \text{and} \quad f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Use the convolution theorem as an alternative way to find the inverse Laplace Transform of

$$\frac{1}{s^2 + 10s + 25}.$$

[3 marks]

Question 9.

Solve the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\cos t,$$

subject to the initial conditions

$$y = 0, \quad \frac{dy}{dt} = 2 \quad \text{at} \quad t = 0.$$

[10 marks]

Question 10.

The aim of this question is to use Laplace Transforms to solve the following inhomogeneous system of ordinary differential equations:

$$\frac{d^2y}{dt^2} + 2y - z = 6\,\delta(t),$$

$$\frac{d^2z}{dt^2} + 3z - 2y = 0.$$

The initial conditions are that $y(0) = y'(0) = z(0) = z'(0) = 0$, and $\delta(t)$ is the unit impulse at $t = 0$.

The definition of the Laplace Transform is given in Questions 2 and 8. You may assume that $\mathcal{L}[y''] = s^2Y(s) - y'(0) - sy(0)$ when $\mathcal{L}[y] = Y(s)$. You may also use the notation that $\mathcal{L}[z] = Z(s)$, and assume that $\mathcal{L}[\sin at] = a/(s^2 + a^2)$.

(a) What is $\mathcal{L}[\delta(t)]$? [1 mark]

(b) Take the Laplace Transform of each of the above equations for y and z, and solve the resulting algebraic equations for Y and Z. [3 marks]

(c) Use partial fractions to simplify your expressions for Y and Z and hence determine both $y(t)$ and $z(t)$. [4 marks]

(d) What are the values of $y'(0)$ and $z'(0)$ from your solutions? Comment on these values in the light of the original initial conditions. [2 marks]