UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2022
Unit Title: Mathematics 2		Unit Code: ME10305
Year: 2021/22	Question Number: 1	Page 1 of 1

Part		Mark
(a)	$\left.\begin{array}{l} y_1 = f \\ y_2 = f' \\ y_3 = f'' \\ y_4 = g \\ y_5 = g' \end{array}\right\} \Rightarrow \begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 - y_1 y_3 \\ y_4' = y_5 \\ y_5' = -y_1 y_5 \end{array}$ $x=0$: $y_1 = y_2 = 0,\ y_4 = 1$; $x \to \infty$: $y_2, y_4 \to 0$	2
(b)	5th order, nonlinear, BVP	2
(c)	$\frac{dy}{y} = \frac{2t}{t^2+1}\,dt \Rightarrow \ln\lvert y\rvert = \ln(t^2+1)+c = \ln A(t^2+1)$ $\Rightarrow y = A(t^2+1)$ $y(0)=1 \Rightarrow A=1.$ Hence $\boxed{y = t^2+1}$	3
(d)	$I.F. = e^{-\int 3/t\,dt} = e^{-3\ln t} = e^{\ln(t^{-3})} = t^{-3}.$ Hence $\frac{1}{t^3}y' - \frac{3y}{t^4} = \frac{4}{t^5}$ $\Rightarrow (y/t^3)' = 4/t^5$ $\Rightarrow y/t^3 = -\frac{1}{t^4} + c$ $\Rightarrow y = ct^3 - \frac{1}{t}$ $y(1)=0 \Rightarrow 0 = c-1 \Rightarrow c=1.$ Hence $\boxed{y = t^3 - \frac{1}{t}}$	3
		Total 10

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(a)	$\mathcal{L}[\sin \omega t] = \int_0^\infty \sin \omega t \, e^{-st} = \text{(Int. by parts)} \text{ (or use } \mathrm{Im}\, e^{j\omega t}) = \boxed{\frac{\omega}{s^2+\omega^2}}$	3
(b)	$\mathcal{L}[f''] = \int_0^\infty f'' e^{-st} dt = [f'][e^{-st}]_0^\infty - [f][-s e^{-st}]_0^\infty + \int_0^\infty [f][s^2 e^{-st}] dt$ $= \boxed{s^2 F - f'(0) - s f(0)}$	2
(c)	$y'' + 9y = 5 \sin 2t$ $\Rightarrow s^2 Y - 5 + 9Y = \frac{10}{s^2+4}$ $\Rightarrow Y = \frac{10}{(s^2+4)(s^2+9)} + \frac{5}{s^2+9}$ $= \frac{2}{s^2+4} - \frac{2}{s^2+9} + \frac{5}{s^2+9}$ $= \frac{2}{s^2+4} + \frac{3}{s^2+9}$ $\Rightarrow \boxed{y = \sin 2t + \sin 3t}$	5

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Part		Mark
(a)	$\begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$ $= 3\left[3\times 8 - 3\right] - 8 = 55$	3
(b)	$\left(\begin{array}{cccc\|c} 2 & 1 & 0 & 0 & 5 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$ $\left(\begin{array}{cccc\|c} 2 & 1 & 0 & 0 & 5 \\ 0 & 3/2 & 1 & 0 & -5/2 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$ $R_2 - \frac{1}{2}R_1$ $\left(\begin{array}{cccc\|c} 2 & 1 & 0 & 0 & 5 \\ 0 & 3/2 & 1 & 0 & -5/2 \\ 0 & 0 & 4/3 & 1 & 5/3 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$ $R_3 - \frac{2}{3}R_2$ $\left(\begin{array}{cccc\|c} 2 & 1 & 0 & 0 & 5 \\ 0 & 3/2 & 1 & 0 & -5/2 \\ 0 & 0 & 4/3 & 1 & 5/3 \\ 0 & 0 & 0 & 5/4 & -5/4 \end{array}\right)$ $R_4 - \frac{3}{4}R_3$ $\Rightarrow d = -1$ $\Rightarrow c = 2$ $\Rightarrow b = -3$ $\Rightarrow a = 4$ $\Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \\ -1 \end{pmatrix}$	7

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Part		Mark
	$y = ax^2 + b$. Hence $S = \sum_{n=1}^{N} (ax_i^2 + b - y_i)^2$	
	$0 = \frac{\partial S}{\partial a} = 2\sum_{n=1}^{N} x_i^2(ax_i^2 + b - y_i) = 2\left[a\sum x_i^4 + b\sum x_i^2 - \sum y_i x_i^2\right]$	
	$0 = \frac{\partial S}{\partial b} = 2\sum_{n=1}^{N} (ax_i^2 + b - y_i) = 2\left[a\sum x_i^2 + bN - \sum y_i\right]$	
	$\Rightarrow \begin{pmatrix} \sum x_i^4 & \sum x_i^2 \\ \sum x_i^2 & N \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i x_i^2 \\ \sum y_i \end{pmatrix}$	3
	MUST HAVE THIS DERIVATION TO GAIN FULL MARKS	
	We find that $\sum x_i^4 = 354$ $\quad \sum y_i x_i^2 = 352.22$	
	$\sum x_i^2 = 30$ $\quad \sum y_i = 30.25$	
	$N = 5$	
	$\Rightarrow \begin{pmatrix} 354 & 30 \\ 30 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 352.22 \\ 30.25 \end{pmatrix}$	
	$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{870}\begin{pmatrix} 5 & -30 \\ -30 & 354 \end{pmatrix}\begin{pmatrix} 352.22 \\ 30.25 \end{pmatrix}$	
	$= \begin{pmatrix} 0.9811 \\ 0.1631 \end{pmatrix}$	
	$\Rightarrow \boxed{y = 0.9811x^2 + 0.1631}$	4
	Through the origin $\Rightarrow b = 0$	
	$\Rightarrow a = \frac{352.22}{354} = 0.9950$. So $\boxed{y = 0.9950x^2}$	3
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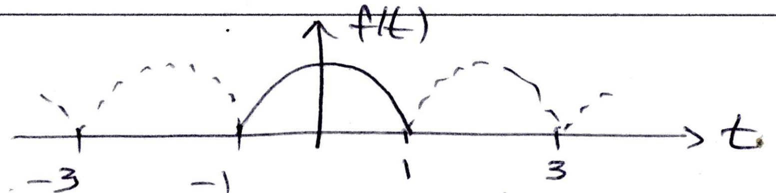
Part		Mark

(a.)



(2)

Even $\Rightarrow B_n = 0$. $\quad A_0 = \int_{-1}^{1}(1-t^2)\,dt = 2\int_0^1 (1-t^2)\,dt = 4/3$

$$A_n = 2\int_0^1 (1-t^2)\cos n\pi t\,dt$$

$$= 2\left[\cancel{(1-t^2)\left(\frac{\sin n\pi t}{n\pi}\right)} - (-2t)\left(-\frac{\cos n\pi t}{n^2\pi^2}\right) + \cancel{(-2)\left(\frac{-\sin n\pi t}{n^3\pi^3}\right)}\right]_0^1$$

$$= \frac{-4\cos n\pi}{n^2\pi^2} = \frac{4}{n^2\pi^2}(-1)^{n+1}$$

$$\Rightarrow \boxed{f(t) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi t}$$

(4)

(b)

$$y'' + 4y = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi t$$

$$\Rightarrow y_{PI} = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2(4-n^2\pi^2)}\cos n\pi t$$

(3)

The 1st & 2nd derivatives will be continuous

(1)

Total	10

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Outline Solution to Examination Question

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Part		Mark
(a)	$x = 1, 4$	1
(b)	Either $x_{n+1} = \dfrac{x_n^2+4}{5}$ or $x_{n+1} = \sqrt{5x_n - 4}$ $x_0 = 1.1$, $x_1 = 1.042$, $x_2 = 1.017\,153$, $x_3 = 1.006\,920$ Converges Use 6DPs $x_0 = 1.1$, $x_1 = 1.224\,745$, $x_2 = 1.457\,300$, $x_3 = 1.812\,871$ Diverges	2
(c)	$x_{n+1} = \dfrac{(1+\varepsilon)^2+4}{5} = \dfrac{5+2\varepsilon+\dots}{5} = 1+\frac{2}{5}\varepsilon \dots$ Conv or $x_{n+1} = [5+5\varepsilon-4]^{1/2} = (1+5\varepsilon)^{1/2} \approx 1+\frac{5}{2}\varepsilon+\dots$ Div	2
(d)	$x_{n+1} = x_n - \dfrac{x_n^2-5x_n+4}{2x_n-5} = \dfrac{x_n^2-4}{2x_n-5}$ $x_0 = 1.1$ $\Rightarrow x_1 = 0.996\,429$ $x_2 = 0.999\,996$ Conv $\to 1$	2
(e)	$x_{n+1} = \dfrac{(1+\varepsilon)^2-4}{2(1+\varepsilon)-5} = \dfrac{-3+2\varepsilon+\varepsilon^2}{-3+2\varepsilon}$ $= 1 + \dfrac{\varepsilon^2}{-3+2\varepsilon} \approx 1 - \dfrac{\varepsilon^2}{3}$ Quad. conv.	3
	Total	10

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Part		Mark

(a) $0 = |A - \lambda I| = \begin{vmatrix} -3-\lambda & 1 & 0 \\ 2 & -3-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{vmatrix}$

$= (-3-\lambda)\left[(-3-\lambda)^2 - 2\right] - 1 \times 2(-3-\lambda)$

$= (-3-\lambda)\left[(-3-\lambda)^2 - 4\right] \quad \Rightarrow \lambda = -3 \text{ or } (\lambda+3)^2 = 4$

$\Rightarrow \boxed{\lambda = -3, -3 \pm 2 = -1, -3, -5}$

$\lambda = -1 \Rightarrow \begin{pmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\lambda = -3 \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = -5 \Rightarrow \begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = C\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (6)

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + Ce^{-5t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (2)

(c) Clearly we need $A = C = 1$, $B = 0$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + e^{-5t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$ (2)

Total 10

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Part		Mark
(a)	$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$ $\quad$ $\mathcal{L}[t] = 1/s^2$ Using I. by P.	2
(b)	$\mathcal{L}[f(t)e^{-at}] = \int_0^\infty f(t)e^{-at}e^{-st}\,dt = \int_0^\infty f(t)e^{-(s+a)t}\,dt$ $= F(s+a)$.	2
(c)	$\frac{1}{s^2+10s+25} = \frac{1}{(s+5)^2}$ So $\mathcal{L}[\underline{te^{-5t}}] = 1/(s+5)^2$ Ans.	3
(d)	$\frac{1}{(s+5)^2} = \frac{1}{s+5} \times \frac{1}{s+5}$ $\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s+5}\times\frac{1}{s+5}\right] = e^{-5t} * e^{-5t}$ $= \int_0^t e^{-5\tau}e^{-5(t-\tau)}\,d\tau$ $= \int_0^t e^{-5t}\,d\tau = e^{-5t}\int_0^t d\tau$ $= te^{-5t}$	3
	Total	10

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Part		Mark

$$\underbrace{y'' + 2y' + y}_{\lambda = -1, -1} = \underbrace{2\cos t}_{\lambda = \pm j}$$

$$\Rightarrow y_{CF} = (A + Bt)e^{-t}$$

Let $y_{PI} = C\cos t + D\sin t$

$$\Rightarrow -2C\sin t + 2D\cos t = 2\cos t$$

$$\Rightarrow C = 0, D = 1$$

$$\Rightarrow y_{GEN} = (A + Bt)e^{-t} + \sin t$$

$y(0) = 0 \Rightarrow A = 0$

$y'(0) = 2$: $\quad y'_{GEN} = (\cancel{-A} - Bt + B)e^{-t} + \cos t$

$$\Rightarrow 2 = B + 1 \quad \Rightarrow B = 1$$

$$\Rightarrow \boxed{y = te^{-t} + \sin t}$$

Total 10

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Part		Mark
(a)	$\mathcal{L}[\delta(t)] = 1$	1

(b) $(s^2+2)Y - Z = 6 \qquad (s^2+3)Z - 2Y = 0.$

$$\Rightarrow \begin{pmatrix} s^2+2 & -1 \\ -2 & s^2+3 \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} Y \\ Z \end{pmatrix} = \frac{1}{s^4+5s^2+4}\begin{pmatrix} s^2+3 & 1 \\ 2 & s^2+2 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6(s^2+3)/(s^2+1)(s^2+4) \\ 12/(s^2+1)(s^2+4) \end{pmatrix}$$

Mark: 3

Other routes through this!

(c)

$$= \begin{pmatrix} 4/(s^2+1) + \dfrac{2}{(s^2+4)} \\ \dfrac{4}{s^2+1} - \dfrac{4}{s^2+4} \end{pmatrix}$$

$$\Rightarrow y = 4\sin t + \sin 2t$$

$$z = 4\sin t - 2\sin 2t$$

Mark: 4

(d) $y'(0) = 6 \qquad z'(0) = 0$

The nonzero value of $y'(0)$ is due to the presence of $6\delta(t)$ as the forcing term for the y-equation

Mark: 2

	Total	10