University of Bath

DEPARTMENT OF MECHANICAL ENGINEERING

ME10305

MATHEMATICS 2

May 2021
2 Hours

Each question carries 10 marks.

The examination consists of TEN questions.
All questions should be attempted.

The marks shown against each **part** *of a question are for guidance only.*

Candidates are permitted to use University-approved calculators.

When uploading the script:

(i) please put the solutions in numerical order,
(ii) please ensure that all pages are in portrait orientation and
(iii) please ensure that there is a strong contrast between the writing and the page.

Question 1.

(a) Write the following system of ordinary differential equations for $f(x)$, $g(x)$ and $h(x)$ and their boundary conditions in first order form:

$$\frac{df}{dx} = g, \qquad \frac{d^2g}{dx^2} + fg - h = 0, \qquad \frac{d^3h}{dx^3} + e^{-x}f + \frac{1}{2}f\frac{d^2h}{dx^2} = g,$$

where the boundary conditions are that,

$$f(0) = 1, \quad h(0) = 0, \quad \frac{dh}{dx}(0) = 1, \quad \frac{d^2h}{dx^2}(0) = -5, \quad \frac{d^2h}{dx^2}(1) = 0.$$ [3 marks]

(b) Classify both the system of equations given in part (a) and its boundary conditions with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem. [2 marks]

(c) Use the technique of separation of variables to solve the equation,

$$\frac{dy}{dx} = y\cos 2x \qquad \text{subject to} \quad y(0) = 2.$$ [2 marks]

(d) Use the method of Integrating Factors to solve the equation,

$$x\frac{dy}{dx} + (1+x)y = (1+\cos x)e^{-x} \qquad \text{subject to} \quad y(\pi) = 2.$$ [3 marks]

Question 2.

The Laplace Transform of $f(t)$ is given by, $F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt.$

(a) Write down the following transforms: $\mathcal{L}[e^{-at}]$ and $\mathcal{L}[\cos\omega t]$ and also state the s-shift theorem. [2 marks]

(b) Find a linear factor of s^3+s-10 by inspection and hence factorise this cubic polynomial in s. [2 marks[

(c) Use partial fractions to simplify $\dfrac{3s+1}{s^3+s-10}$, and hence determine its inverse Laplace Transform using the results from part (a) above. [6 marks]

Question 3.

(a) Find the determinant of the matrix,

$$A = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 2 & 2 & 1 & -1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

[4 marks]

(b) Use Gaussian Elimination to solve the following system of simultaneous equations:

$$\begin{pmatrix} 1 & 4 & 1 \\ 3 & 16 & 5 \\ -1 & 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ -1 \end{pmatrix}.$$

[6 marks]

Workings will need to be shown for both parts of this question.

Question 4.

(a) The function, $f(t)$, has a period equal to 1 and it is given by $f(t) = t - \frac{1}{2}$ within the range, $0 < t < 1$. Sketch the function in the range $-2 < t < 2$. [2 marks]

(b) Find the Fourier series representation of the function, $f(t)$, given in part (a). (The definitions of the Fourier coefficients are given below.) [4 marks]

(c) Using the result of part (b) find the particular integral of the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 4y = f(t).$$

[3 marks]

(d) Clearly, $y(t)$ will be continuous at $t = 1$, but how many of its derivatives will also be continuous at $t = 1$ before a discontinuous one is encountered? [1 mark]

You may use the following definitions of the Fourier Coefficients,

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \cos 2n\pi t + B_n \sin 2n\pi t\Big],$$

where

$$A_n = 2\int_0^1 f(t)\,\cos 2n\pi t\,dt \qquad n = 0, 1, \cdots, \infty,$$

$$B_n = 2\int_0^1 f(t)\,\sin 2n\pi t\,dt \qquad n = 1, 2, \cdots, \infty.$$

Question 5.

The following data have been obtained from an experiment.

x_i	0	0.2	0.4	0.6	0.8	1
y_i	0.011	0.079	0.253	0.467	0.788	1.212

(a) It is proposed to fit the quadratic curve, $y = ax^2 + bx$, to this data using the method of least squares. Derive the equation which will be used to determine a and b. [4 marks]

(b) Hence find the curve of best fit. [6 marks]

Question 6.

The aim for this question is to employ some iteration schemes to solve the equation, $x^3 + x = 2$.

(a) Use a suitable sketch to determine how many real roots this equation has. [2 marks]

(b) Write down the two possible *ad hoc* iteration schemes for solving this cubic equation numerically. Beginning with the initial iterate, $x_0 = 1.1$, use each of the formulae to evaluate the next four iterates. Retain six decimal places in your computations. [2 marks]

(c) Analyse the rates of convergence/divergence of each of these *ad hoc* schemes by setting $x_n = 1 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating x_{n+1}. [2 marks]

(d) Write down the Newton-Raphson scheme for the given cubic equation . Use this formula to evaluate the next three iterates beginning this time with $x_0 = 1.2$. Retain six decimal places in your computations. [2 marks]

(e) Analyse the approach to the root $x = 1$ for the Newton-Raphson scheme in part (d) by setting $x_n = 1 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating x_{n+1}. [2 marks]

Question 7.

(a) Find all the eigenvalues and eigenvectors of the matrix, $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. [6 marks]

(b) Use the result of part (a) to write down the general solution of the following system of ordinary differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

[2 marks]

(c) Hence find the solution of that system which satisfies the initial condition that $x = 0$, $y = 0$ and $z = 3$ at $t = 0$. [2 marks]

Question 8.

The Laplace Transform of $f(t)$ is defined according to,

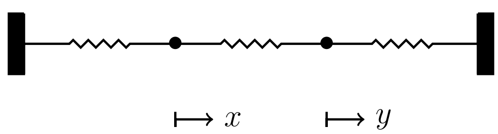
$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt.$$

You are allowed to assume the following results without proof:

$$\mathcal{L}\left[f''(t)\right] = s^2 \mathcal{L}\left[f(t)\right] - f'(0) - sf(0),$$

$$\mathcal{L}\left[\sin \omega t\right] = \frac{\omega}{s^2 + \omega^2}.$$

Two masses which are connected by springs to one another and to unmovable supports are illustrated in the following diagram. The values $x(t)$ and $y(t)$ denote the distances from equilibrium of the the masses and the arrows indicate their directions of movement. Initially both masses have a zero displacement and a unit velocity. The aim of the question is to use Laplace Transform methods to determine the subsequent motion of the masses.



The governing equations are,

$$\frac{d^2x}{dt^2} + 5x - 4y = 0,$$

and

$$\frac{d^2y}{dt^2} - 4x + 5y = 0.$$

The initial conditions are that $x = y = 0$ and $x' = y' = 1$ at $t = 0$.

Find the solutions for x and y using Laplace Transform methods. [10 marks]

Question 9.

(a) Solve the ordinary differential equations,

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = 3,$$

subject to the initial conditions,

$$y(0) = 1, \quad y'(0) = 1 \quad \text{and} \quad \frac{d^2y}{dt^2}(0) = 2.$$

[5 marks]

(b) Solve the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 4y = 4\cos 2t,$$

subject to the initial conditions,

$$y(0) = 0 \quad \text{and} \quad \frac{dy}{dt}(0) = 2.$$

[5 marks]

Question 10.

The aim of this question is to develop a least squares theory to fit two functions to a given function rather than to a set of data points.

(a) It is proposed to fit $y = a + b\cos x$ to a general function $f(x)$ in the range $-\pi \le x \le \pi$. The residual, $r(x)$, is defined as,

$$r(x) = f(x) - a - b\cos x.$$

Form the integral of the square of the residual in the range, $-\pi \le x \le \pi$, and minimise the value of this integral with respect to both a and b in order to find integral expressions for a and b. [5 marks]

(b) Are the formulae for a and b familiar? If so, then what do they remind you of? [2 marks]

(c) Hence find the curve of best fit to the function $f(x) = x^2$. [3 marks]