## REMOTE ASSESSMENT RUBRIC/COVER SHEET

**University of Bath**
**Department of Mechanical Engineering**
**ME10305 Mathematics 2**

**Assessment available from:** 9am Monday 11th May 2020
**Latest submission time:** 9.30am Tuesday 2nd June 2020
*All timings are given in British Summer Time (BST)*

Between these times you must complete and submit your completed assessment. This assessment is designed to take approximately 2 hours to complete.

This is an open book examination. You may refer to your own course and revision notes and look up information in offline or online resources, for example textbooks or online journals. However, you may not communicate with any person or persons about this assessment before the submission deadline unless explicitly permitted to do so in the instructions below. When you submit your assignment, you will be asked to agree to an academic integrity declaration and confirm the work is your own. It is expected that you will have read and understood the Regulations for Students and your programme handbook, including the references to and penalties for unfair practices such as plagiarism, fabrication or falsification.

**All questions should be attempted. Each of the 10 questions carries 10 marks.**

**The marks shown against each part of a question are for guidance only.**

**Please write your answers on the solution sheets which form the last three pages of the present document. Please do not include your workings-out in the submission.**

**The University Formula Book may be found here.**

**Submitting your assessment:** When you have completed this assessment, you must submit your work in PDF format as a **single file**, uploaded to the Moodle submission point relating to this assessment. Your PDF document should be **legible, with all pages upright and in order.** If you do not have a scanner available, please follow the instructions for creating a PDF file on a mobile device that may be found here. Additional guidance on how to submit your assessment is available here.

**Please name your pdf file in the following way: FamilyNameFirstNameStudentIDUnitCode.pdf with no gaps. So if I were submitting a script and my StudentID were 31415926, then my file would be named: ReesAndrew31415926ME10305.pdf**

## Question 1.

(a) Write the following pair of ordinary differential equations for $f(x)$ and $g(x)$ and their boundary conditions in first order form:

$$\frac{d^3 f}{dx^3} + \frac{3}{4} f \frac{d^2 f}{dx^2} - \frac{1}{2}\left(\frac{df}{dx}\right)^2 + g = 0, \qquad \frac{d^2 g}{dx^2} + \frac{3}{4} f \frac{dg}{dx} = 0,$$

where the boundary conditions are that,

$$f = \frac{df}{dx} = 0 \quad \text{and} \quad g = 1 \quad \text{at} \quad x = 0$$

while

$$\frac{df}{dx} \to 0 \quad \text{and} \quad g \to 0 \quad \text{as} \quad x \to \infty.$$

[3 marks]

(b) Classify the system of equations given in part (a) and its boundary conditions with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem. [2 marks]

(c) Use the technique of separation of variables to solve the equation,

$$y\frac{dy}{dt} = t(1 + y^2) \qquad \text{subject to} \quad y(0) = 1.$$

[2 marks]

(d) Use the method of Integrating Factors to solve the equation,

$$t\frac{dy}{dt} - (1 + t)y = -e^t \qquad \text{subject to} \quad y(1) = 0.$$

[3 marks]

## Question 2.

The Laplace Transform of $f(t)$ is given by, $F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt.$

(a) Write down the following transforms: $\mathcal{L}[e^{-at}]$ and $\mathcal{L}[\,t^2\,]$. [2 marks]

(b) Use the $s$-shift theorem to determine the inverse Laplace Transform of

$$\frac{1}{s^3 + 3s^2 + 3s + 1}.$$

[4 marks]

(c) Use partial fractions to determine the inverse Laplace Transform of

$$\frac{s}{s^2 + s - 6}.$$

[4 marks]

**Question 3.**

(a) Find the determinant of the matrix,

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & -1 \\ 2 & 2 & 4 & -2 \\ 1 & 2 & 2 & -1 \end{pmatrix}$$

[4 marks]

(b) Use Gaussian Elimination to solve the following system of simultaneous equations:

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & -2 & 5 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix}.$$

[6 marks]

Please provide both the final answer and the upper triangular form which is obtained at the end of the elimination phase of the algorithm.

**Question 4.**

(a) The function, $f(t)$, has a period equal to $2$ and it is given by $f(t) = t - t^3$ within the range, $-1 < t < 1$. Sketch the function in the range $-3 < t < 3$. [2 marks]

(b) Find the Fourier series representation of the function, $f(t)$, given in part (a). (The definitions of the Fourier coefficients are given below.) [4 marks]

(c) Using the result of part (b) find the particular integral of the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 4y = f(t).$$

[3 marks]

(d) Clearly, $y(t)$ will be continuous at $t = 1$, but how many of its derivatives will also be continuous at that point? [1 mark]

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You may use the following definitions of the Fourier Coefficients,

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \cos n\pi t + B_n \sin n\pi t\Big],$$

where

$$A_n = \int_{-1}^{1} f(t)\, \cos n\pi t\, dt \qquad n = 0, 1, \cdots, \infty,$$

$$B_n = \int_{-1}^{1} f(t)\, \sin n\pi t\, dt \qquad n = 1, 2, \cdots, \infty.$$

**Question 5.**

The following data have been obtained from an experiment.

| $x_i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y_i$ | 0.0443 | 2.0973 | 2.2832 | 0.3805 | −3.3893 |

(a) It is proposed to fit the quadratic curve, $y = ax^2 + bx$, to this data using the method of least squares. Write down the equation which will be used to determine $a$ and $b$. [3 marks]

(b) Hence find the line of best fit. [7 marks]

**Question 6.**

(a) The aim for this question is to employ some iteration schemes to solve the equation, $x^3 - 13x + 12 = 0$. By inspection identify one of the roots and hence factorise the cubic. [1 mark]

(b) There are two possible *ad hoc* schemes for solving this cubic equation numerically. Write them down, and use each of the formulae to evaluate the next four iterates beginning with $x_0 = 1.1$. Retain six decimal places in your computations. [3 marks]

(c) Analyse the rates of convergence/divergence of both of these *ad hoc* schemes by setting $x_n = 1 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating $x_{n+1}$. [2 marks]

(d) Write down the Newton-Raphson scheme for the given cubic equation . Use this formula to evaluate the next three iterates beginning this time with $x_0 = 1.2$. [2 marks]

(e) Analyse the approach to the root $x = 1$ for this Newton-Raphson scheme by setting $x_n = 1 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating $x_{n+1}$. [2 marks]

**Question 7.**

(a) Find all the eigenvalues and eigenvectors of the matrix, $\begin{pmatrix} 0 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$ [6 marks]

(b) Use the result of part (a) to write down the general solution of the following system of ordinary differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

[3 marks]

(c) Hence find the solution of that system which satisfies the initial condition that $x = 0$, $y = 3$ and $z = 0$ at $t = 0$. [1 mark]

**Question 8.**

The Laplace Transform of $f(t)$ is defined according to,

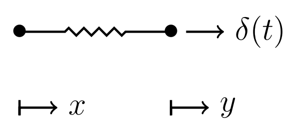
$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt.$$

You are allowed to assume the following results without proof:

$$\mathcal{L}\left[f''(t)\right] = s^2 \mathcal{L}\left[f(t)\right] - f'(0) - s f(0),$$

$$\mathcal{L}\left[t\right] = \frac{1}{s^2} \qquad \text{and} \qquad \mathcal{L}\left[\sin \omega t\right] = \frac{\omega}{s^2 + \omega^2}.$$

(a) Two masses which are connected by a spring are illustrated in the following diagram. The values $x(t)$ and $y(t)$ denote the distances from equilibrium of the two masses and the arrows their direction of movement. The masses are initially at rest and each has a zero displacement from equilibrium at $t = 0$. The aim of the question is to determine the motion of the masses after the right hand one has been subjected to a unit impulse at $t = 0$.




The governing equations are,

$$\frac{d^2x}{dt^2} + x - y = 0,$$

and

$$\frac{d^2y}{dt^2} - x + y = \delta(t),$$

where $\delta(t)$ is the unit impulse. The initial conditions are that $x = x' = y = y' = 0$ at $t = 0$. Find the solutions for $x$ and $y$ using Laplace Transform methods. [7 marks]

(b) From these solutions, find the initial velocity for each of the masses and hence sketch the variations of $x$ and $y$ with time. Why does one of initial velocities not agree with the imposed initial condition? [3 marks]

**Question 9.**

(a) Solve the ordinary differential equations,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 2t,$$

subject to the initial conditions,

$$y(0) = 0 \quad \text{and} \quad \frac{dy}{dt}(0) = 0.$$

[5 marks]

(b) Solve the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 13\cos 2t,$$

subject to the initial conditions,

$$y(0) = 0 \quad \text{and} \quad \frac{dy}{dt}(0) = 3.$$

[5 marks]

**Question 10.**

(a) Use an Integrating Factor to solve the ordinary differential equation,

$$r\frac{dy}{dr} + 2y = 4\ln r,$$

subject to the initial condition, $y(1) = 0$. [5 marks]

(b) Solve the ordinary differential equation which is given in part (a) by first making the substitution, $r = e^x$, to transform it into a differential equation for $y$ in terms of $x$. [5 marks]

Name:

Email: Student ID:

Q1a. The equations become: [2 marks]

The boundary conditions become: [1 mark]

Q1b. Classification: [2 marks]

Q1c. Solution: [2 marks]

Q1d. Integrating Factor: [1 mark]

Solution: [2 marks]

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Q2a. Solutions: [2 marks]

Q2b. Solution: [4 marks]

Q2c. Solution: [4 marks]

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Q3a. Determinant: [4 marks]

Q3b. Upper triangular form: [4 marks]

Final solution: [2 marks]

Q4a. Sketch: [2 marks]

Q4b. Fourier Series: [4 marks]

Q4c. Solution: [3 marks]

Q4d. Answer: [1 mark]

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Q5a. Formula being used: [3 marks]

Q5b. Numerical solution: [7 marks]

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Q6a. Factorisation: [1 mark]

Q6b. Ad hoc schemes: [1 mark]

Iterations: [2 marks]

Q6c. $x_{n+1} =$ [1 mark]

$x_{n+1} =$ [1 mark]

Q6d. Iteration scheme: [1 mark]

Iterates: [1 mark]

Q6e. $x_{n+1} =$ [2 marks]

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Q7a. Eigenvalues and eigenvectors: [6 marks]

Q7b. General solution: [3 marks]

Q7c. Final solution: [1 marks]

Q8a. Solutions: [7 marks]

Q8b. Sketch: [1 mark]

Reason: [2 marks]

Q9a. Solution: [5 marks]

Q9b. Solution: [5 marks]

Q10a. Integrating Factor: [2 marks]

Solution: [3 marks]

Q10b. Equation for $y$ in terms of $x$: [2 marks]

Solution in terms of $x$: [2 marks]

Solution in terms of $r$: [1 mark]