Each question carries 10 marks.

The examination consists of TEN questions.
All questions should be attempted.

The marks shown against each **part** *of a question are for guidance only.*

Candidates are permitted to use University-approved calculators.

Question 1.

(a) Write the following ordinary differential equation and its boundary conditions in first order form:

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 1,$$

where the boundary conditions are that,

$$y = \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0 \qquad \text{and} \qquad y = 1 \quad \text{at} \quad x = 1.$$ [2 marks]

Classify this equation and its boundary conditions with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem (giving reasons). [2 marks]

(b) Use the method of Integrating Factors to solve the equation,

$$t\frac{dy}{dt} = y - t^2 \qquad \text{subject to} \quad y(1) = 0.$$ [3 marks]

(c) Use the technique of separation of variables to solve the equation,

$$t\frac{dy}{dt} = y - t^2 y \qquad \text{subject to} \quad y(1) = 1.$$ [3 marks]

Question 2.

The Laplace Transform of $f(t)$ is defined according to

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt.$$

(a) Use the above definition of the Laplace Transform to find the Laplace Transform of the function $\sin \omega t$. [3 marks]

(b) Use the above definition of the Laplace Transform to find the transform of the function, $f''(t)$, in terms of $F(s)$, $f(0)$ and $f'(0)$. [3 marks]

(c) Hence solve the second order ordinary differential equation,

$$x'' + 4x = 3\sin t,$$

subject to the initial conditions, $x(0) = 0$ and $x'(0) = 3$. [4 marks]

Question 3.

(a) Find the determinant of the matrix,

$$A = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \end{pmatrix}$$

[4 marks]

(b) Use Gaussian Elimination to solve the following system of simultaneous equations:

$$\begin{pmatrix} -1 & 2 & 1 \\ -2 & 5 & 3 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -3 \end{pmatrix}.$$

[6 marks]

[Note: Full marks will only be awarded for a correct answer which has used the Gaussian Elimination method in its precise form, i.e. where the matrix has been reduced to upper triangular form.]

Question 4.

(a) The function, $f(t)$, has a period equal to 2 and it is given by $f(t) = t^2$ within the range, $-1 < t < 1$. Sketch the function in the range $-3 < t < 3$ and find its Fourier series representation. (The definitions of the Fourier coefficients are given below.) [6 marks]

(b) Hence find the particular integral of the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 4y = f(t).$$

Clearly, $y(t)$ will be continuous at $t = 1$, but how many of its derivatives will also be continuous at that point? [4 marks]

You may use the following definitions of the Fourier Coefficients,

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} \Big[A_n \cos n\pi t + B_n \sin n\pi t \Big],$$

where

$$A_n = \int_{-1}^{1} f(t)\, \cos n\pi t \, dt \qquad n = 0, 1, \cdots, \infty,$$

$$B_n = \int_{-1}^{1} f(t)\, \sin n\pi t \, dt \qquad n = 1, 2, \cdots, \infty.$$

Question 5.

The following data have been obtained from an experiment.

x_i	0	1	2	3	4
y_i	1.943	2.575	3.912	6.506	10.091

It is proposed to fit the quadratic line, $y = ax^2 + c$, to this data using the method of least squares. Develop the theory, and hence find the line of best fit. [4,6 marks]

Question 6.

(a) The aim for this question is to employ some iteration schemes to solve the equation, $x^2 - 5x + 6 = 0$. First solve the equation analytically. [1 mark]

(b) There are two possible *ad hoc* schemes for solving this quadratic equation numerically. Write one of them down, and use this formula to evaluate the first three iterates beginning with $x_0 = 2.1$. [2 marks]

(c) Analyse the rate of convergence of this *ad hoc* scheme by setting $x_n = 2 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating x_{n+1}. [2 marks]

(d) Write down the Newton-Raphson scheme for this quadratic. Use this formula to evaluate the first two iterates beginning again with $x_0 = 2.1$. [2 marks]

(e) Analyse the approach to the limit $x_\infty = 2$ for this Newton-Raphson scheme by setting $x_n = 2 + \epsilon$ where $|\epsilon| \ll 1$, and by evaluating x_{n+1}. [3 marks]

Question 7.

(a) Find all the eigenvalues of the matrix,

$$\begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 1 \end{pmatrix},$$

and determine their corresponding eigenvectors. [6 marks]

(b) Use the result of part (a) to write down the general solution of the following system of ordinary differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

[4 marks]

Question 8.

The Laplace Transform of $f(t)$ is defined according to,

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt.$$

(a) Use the definition of the Laplace Transform to find both $\mathcal{L}[e^{-at}]$ and $\mathcal{L}[\,t\,]$. [2 marks]

(b) Use the definition of the Laplace Transform to prove the s-shift theorem

$$\mathcal{L}[f(t)e^{-at}] = F(s+a)$$

where $\mathcal{L}[f(t)] = F(s)$. [2 marks]

(c) Use the s-shift theorem to determine the inverse Laplace Transform of

$$\frac{1}{s^2+6s+9}.$$

[3 marks]

(d) The convolution theorem is $\mathcal{L}[f * g] = F(s)G(s)$, where

$$\mathcal{L}[f(t)] = F(s), \quad \mathcal{L}[g(t)] = G(s) \quad \text{and} \quad f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Use the convolution theorem as an alternative way to find the inverse Laplace Transform of

$$\frac{1}{s^2+6s+9}.$$

[3 marks]

Question 9.

(a) Use the substitution $y(t) = z(t)e^{-t}$ to transform the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 6te^{-t}$$

into an ordinary differential equation for $z(t)$. [5 marks]

(b) Solve this equation for z and hence write down the general solution for $y(t)$. [2 marks]

(c) What is the specific solution which satisfies the initial conditions, $y(0) = 0$ and $y'(0) = 1$? [3 marks]

Question 10.

The aim of this question is to use Laplace Transforms to solve the following inhomogeneous system of ordinary differential equations:

$$\frac{d^2y}{dt^2} - z = 0,$$

$$\frac{d^2z}{dt^2} + 5z + 4y = 3\delta(t).$$

The initial conditions are that $y(0) = y'(0) = z(0) = z'(0) = 0$, and $\delta(t)$ is the unit impulse at $t = 0$.

The definition of the Laplace Transform is given in Questions 2 and 8. You may assume that $\mathcal{L}[y''] = s^2Y(s) - y'(0) - sy(0)$ when $\mathcal{L}[y] = Y(s)$. You may also use the notation that $\mathcal{L}[z] = Z(s)$, and assume that $\mathcal{L}[\sin at] = a/(s^2 + a^2)$.

(a) What is $\mathcal{L}[\delta(t)]$? [1 mark]

(b) Take the Laplace Transform of each of the above equations for y and z, then eliminate Z between the resulting algebraic equations for Y and Z in order to find Y. [3 marks]

(c) Use partial fractions to determine $y(t)$, and hence find $z(t)$. [3 marks]

(d) What are the values of $y'(0)$ and $z'(0)$ from your solutions? Comment on these values in the light of the original initial conditions. [3 marks]