UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination ~~n~~ Question

Examiner: Dr D A S Rees		Date May 2019
Unit Title: Mathematics 2		Unit Code: ME10305
Year: 2018/19	Question Number: 1	Page 1 of 1

Part		Mark
(a)	Let $y_1 = y, \quad y_2 = y', \quad y_3 = y'' \quad \Rightarrow \quad y_1' = y_2, \quad y_2' = y_3, \quad y_3' = 1 - 2y_1y_2 - y_1y_3$	
	$y_1 = y_2 = 0$ at $x = 0 \qquad y_1 = 1$ at $x = 1$	2
	3rd order, nonlinear BVP	2
(b)	Correct 1st order form: $y' - \frac{1}{t}y = -t$	
	I.F. is $e^{\int -\frac{1}{t}dt} = e^{-\ln t} = 1/t$	
	$\Rightarrow \frac{1}{t}y' - \frac{1}{t^2}y = -1 \quad \Rightarrow (y/t)' = -1$	
	$\Rightarrow y/t = c - t \quad \Rightarrow y = ct - t^2$.	
	I.C. is $y(1) = 0 \quad \Rightarrow c = 1 \quad \Rightarrow \boxed{y = t - t^2}$	3
(c)	$t\frac{dy}{dt} = (1-t^2)y \quad \Rightarrow \frac{dy}{y} = \left(\frac{1}{t} - t\right)dt$	
	$\Rightarrow \ln\lvert y\rvert = \ln\lvert t\rvert - t^2/2 + c$	
	$\Rightarrow y = Ate^{-t^2/2}$.	
	I.C. is $y(1) = 1 \quad \Rightarrow A = e^{1/2}$	
	Hence $\boxed{y = t\,e^{(1-t^2)/2}}$.	3
	Total	10

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(a)	$\mathcal{L}[\sin\omega t] = \int_0^\infty \sin\omega t\, e^{-st}\,dt = \mathrm{Im}\int_0^\infty e^{j\omega t} e^{-st}\,dt$	
	$= \mathrm{Im}\left[\frac{1}{s-j\omega}\right] = \mathrm{Im}\left[\frac{s+j\omega}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2}$	3
	[Or use integration by parts]	
(b)	$\mathcal{L}[f''] = \int_0^\infty f'' e^{-st}\,dt$	
	$= [f'][e^{-st}]_0^\infty - [f][-se^{-st}]_0^\infty + \int_0^\infty [f][s^2 e^{-st}]\,dt$	
	$= s^2F - f'(0) - sf(0)$. BOOKWORK.	3
(c)	$x'' + 4x = 3\sin t \xrightarrow{LT} s^2X - 3 + 4X = \frac{3}{s^2+1}$	
	$\Rightarrow X = \frac{3}{s^2+4} + \frac{3}{(s^2+1)(s^2+4)}$	
	$= \frac{3}{s^2+4} + \frac{1}{s^2+1} - \frac{1}{s^2+4}$	
	$= \frac{2}{s^2+4} + \frac{1}{s^2+1}$	
	$\Rightarrow x = \sin 2t + \sin t$.	4
	Total	10

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Part		Mark
(a)	Many ways of doing this.	
	$\begin{vmatrix}1&3&-1&1\\2&0&1&1\\2&2&-1&1\\2&4&-2&2\end{vmatrix} = \begin{vmatrix}1&3&-1&1\\1&-3&2&0\\1&-1&0&0\\0&-2&0&0\end{vmatrix} \begin{matrix}\\R_2-R_1\\R_3-R_1\\R_4-2R_1\end{matrix} = -(1)\begin{vmatrix}1&-3&2\\1&-1&0\\0&-2&0\end{vmatrix}$	
	$= -2\begin{vmatrix}1&-1\\0&-2\end{vmatrix} = 4$	4
(b)	$\left(\begin{array}{ccc\|c}-1&2&1&3\\-2&5&3&10\\1&-3&0&-3\end{array}\right) \longrightarrow \left(\begin{array}{ccc\|c}-1&2&1&3\\0&1&1&4\\0&-1&1&0\end{array}\right) \begin{matrix}\\R_2-2R_1\\R_3+R_1\end{matrix}$	
	$\longrightarrow \left(\begin{array}{ccc\|c}-1&2&1&3\\0&1&1&4\\0&0&2&4\end{array}\right) \begin{matrix}\\\\R_3+R_2\end{matrix}$	4
	So $2z=4 \Rightarrow z=2$	
	$y+z=4 \Rightarrow y=2$	
	$-x+2y+z=3 \Rightarrow x=3$	2
	Total	10

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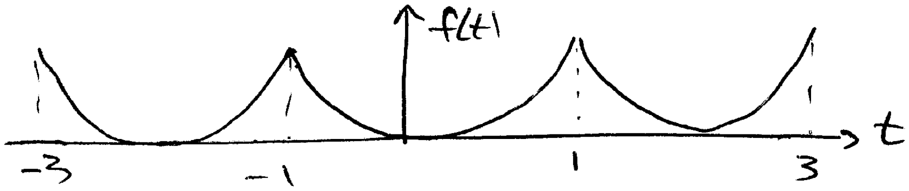
Part		Mark
(a)		1
	$A_0 = \int_{-1}^{1} t^2\,dt = 2\int_0^1 t^2\,dt = 2/3$	
	$B_n = 0$ because $f(t)$ is even.	
	$A_n = \int_{-1}^{1} t^2 \cos n\pi t\,dt = 2\int_0^1 t^2 \cos n\pi t\,dt$	
	$= 2\left[(t^2)\left[\cancel{\frac{\sin n\pi t}{n\pi}}\right]_0^1 - [2t]\left[-\frac{\cos n\pi t}{n^2\pi^2}\right] + [2]\left[\cancel{-\frac{\sin n\pi t}{n^3\pi^3}}\right]_0^1 \right]$	
	$= \frac{4}{n^2\pi^2}\left[t \cos n\pi t\right]_0^1 = \frac{4(-1)^n}{n^2\pi^2}$	
	$\Rightarrow \boxed{f(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos n\pi t.}$	5
	$y'' + 4y = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos n\pi t$	
	$\Rightarrow \boxed{y_{PI} = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos n\pi t}{n^2\pi^2(4 - n^2\pi^2)}}$	3
	y, y' & y'' will be continuous.	1

Total	10

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Outline Solution to Examination Question

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Part		Mark
.	Use $S = \sum (y_n - a x_n^2 - c)^2$ Need $0 = \frac{\partial S}{\partial a} = -2\sum x_n^2 (y_n - a x_n^2 - c)$ & $0 = \frac{\partial S}{\partial c} = -2\sum (y_n - a x_n^2 - c)$ $\Rightarrow \begin{pmatrix} \sum x_n^4 & \sum x_n^2 \\ \sum x_n^2 & N \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \sum y_n x_n^2 \\ \sum y_n \end{pmatrix}$.	4
	Using the calculator $\sum x_n^2 = 30 \qquad \sum x_n^4 = 354 \qquad N = 5$ $\sum y_n = 25.027 \qquad \sum x_n^2 y_n = 238.233$	2
	$\Rightarrow \begin{pmatrix} 354 & 30 \\ 30 & 5 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 238.233 \\ 25.027 \end{pmatrix}$ $\Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \frac{1}{870} \begin{pmatrix} 5 & -30 \\ -30 & 354 \end{pmatrix} \begin{pmatrix} 238.233 \\ 25.027 \end{pmatrix}$ $= \begin{pmatrix} 0.5062 \\ 1.9685 \end{pmatrix}$ So $y = 1.9685 + 0.5062 x^2$	4

Total	10

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Part		Mark

(a) $(x-2)(x-3)=0 \implies x = 2, 3$

Mark: 1

(b) Scheme 1: $x_{n+1} = (5x_n - 6)^{1/2}$

$$2.1 \longrightarrow 2.121320 \longrightarrow 2.146300 \longrightarrow 2.175201$$

Diverging slowly.

Scheme 2: $x_{n+1} = \frac{1}{5}(x_n^2 + 6)$

$$2.1 \longrightarrow 2.082 \longrightarrow 2.066945 \longrightarrow 2.054452$$

Converging slowly

Mark: 2

(c) Scheme 1: $x_{n+1} = [10 + 5\varepsilon - 6]^{1/2} = (4+5\varepsilon)^{1/2} = 2(1+\tfrac{5}{4}\varepsilon)^{1/2}$

$$= 2(1 + \tfrac{5}{8}\varepsilon + \cdots) = 2 + \tfrac{5}{4}\varepsilon \cdots$$

Scheme 2: $x_{n+1} = \dfrac{(2+\varepsilon)^2 + 6}{5} = \dfrac{10 + 4\varepsilon + \cdots}{5} = 2 + \tfrac{4}{5}\varepsilon + \cdots$

Mark: 2

(d) NR is $x_{n+1} = x_n - \dfrac{(x_n^2 - 5x_n + 6)}{2x_n - 5} = \dfrac{x_n^2 - 6}{2x_n - 5}$

$$2.1 \longrightarrow 1.99875 \longrightarrow 1.999848$$

Converging quickly.

Mark: 2

(e) $$x_{n+1} = \frac{(2+\varepsilon)^2 - 6}{2(2+\varepsilon) - 5} = \frac{4 + 4\varepsilon + \varepsilon^2 - 6}{4 + 2\varepsilon - 5}$$

$$= \frac{-2 + 4\varepsilon + \varepsilon^2}{-1 + 2\varepsilon} = \frac{2(-1+2\varepsilon) + \varepsilon^2}{-1+2\varepsilon}$$

$$= 2 + \frac{\varepsilon^2}{-1+2\varepsilon} \approx 2 - \varepsilon^2.$$

Quadratic convergence.

Mark: 3

Total	10

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Part		Mark
(a)	$0 = \begin{vmatrix} 1-\lambda & -2 & 0 \\ -1 & 1-\lambda & -1 \\ 0 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2-2\right] - 2(1-\lambda)$	
	$= (1-\lambda)\left[(1-\lambda)^2-4\right]$	
	$\Rightarrow \lambda = 1$ and $(1-\lambda)^2 = 4.$	
	$\Rightarrow \lambda = 1, -1, 3.$	3
	$\lambda = 1 \Rightarrow \begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & -1 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	
	$\lambda = -1 \Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	
	$\lambda = 3 \Rightarrow \begin{pmatrix} -2 & -2 & 0 \\ -1 & -2 & -1 \\ 0 & -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = C\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	3
(b)	Setting $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} e^{\lambda t}$ gives the above eigenvalue problem.	
	Hence $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{t} + B\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{-t} + C\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}e^{3t}$	4
	Total	10

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Part		Mark
(a)	$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$. $\mathcal{L}[t] = \int_0^\infty t e^{-st} dt = [t]\left[\frac{e^{-st}}{-s}\right]_0^\infty - [1]\left[\frac{e^{-st}}{s^2}\right]_0^\infty \to \frac{1}{s^2}$.	2
(b)	$\mathcal{L}[f(t)e^{-at}] = \int_0^\infty f(t) e^{-(s+a)t} dt = F(s+a)$	2
(c)	$\frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2}$. Hence $\mathcal{L}[te^{-3t}] = \frac{1}{(s+3)^2}$	3
(d)	Let $F = G = \frac{1}{s+3} \Rightarrow f = g = e^{-3t}$ So $\mathcal{L}^{-1}\left[\frac{1}{s^2+6s+9}\right] = f * g$ $= \int_0^t e^{-3\tau} e^{-3(t-\tau)} dt$ $= e^{-3t}\int_0^t 1\, dt = te^{-3t}$.	3

Total	10

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Part		Mark
(a)	If $y = ze^{-t}$ then $y' = (z'-z)e^{-t}$ & $y'' = (z''-2z'+z)e^{-t}$ $\Rightarrow e^{-t}\left[(z''-2z'+z) + 2(z'-z) + z\right] = 6te^{-t}$ $\Rightarrow z'' = 6t$	5
(b)	$\Rightarrow z' = 3t^2 + c \quad \Rightarrow z = t^3 + ct + d.$ $\Rightarrow y = (t^3 + ct + d)e^{-t}$	2
(c)	$y(0) = 0 \Rightarrow d = 0.$ $y' = (3t^2+c)e^{-t} - (t^3+ct)e^{-t}$ $y'(0) = 1 \Rightarrow c = 1$ Hence $y = (t^3+t)e^{-t}$	3
	Total	10

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Outline Solution to Examination Question

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Part		Mark
(a)	$\mathcal{L}[\delta(t)] = 1$	1
(b)	We get $s^2Y - Z = 0$ $\quad s^2Z + 5Z + 4Y = 3.$ Eliminating Z gives $(s^4+5s^2+4)Y = 3$ $\Rightarrow Y = \dfrac{1}{(s^4+5s^2+4)} = \dfrac{3}{(s^2+4)(s^2+1)}$	3
(c)	$Y = \dfrac{1}{s^2+1} - \dfrac{1}{s^2+4} \Rightarrow y = \sin t - \frac{1}{2}\sin 2t$ Various ways to find $z(t)$. Will use $z = y''$ $\Rightarrow z = -\sin t + 2\sin 2t.$	3
(d)	From above $y' = \cos t - \cos 2t$ $\Rightarrow y'(0) = 0$ ✓ $z' = -\cos t + 4\cos 2t$ $\Rightarrow z'(0) = 3.$ The nonzero value for $z'(0)$ is due to the $3\delta(t)$ forcing term. The presence of $3\delta(t)$ ~~provides~~ increases the momentum by 3 instantaneously.	3

Total	10