Each question carries 10 marks.

The examination consists of TEN questions.
All questions should be attempted.

The marks shown against each **part** *of a question are for guidance only.*

Candidates are permitted to use University-approved calculators.

Question 1.

(a) Write the following ordinary differential equation and its boundary conditions in first order form:

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 1,$$

where the boundary conditions are that,

$$y = \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0 \qquad \text{and} \qquad y = 1 \quad \text{at} \quad x = 1.$$ [2 marks]

Classify this equation and its boundary conditions with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem (giving reasons). [2 marks]

(b) Use the method of Integrating Factors to solve the equation,

$$t\frac{dy}{dt} = y - t^3 \qquad \text{subject to} \quad y(1) = 1.$$ [3 marks]

(c) Use the technique of separation of variables to solve the equation,

$$t\frac{dy}{dt} = y - t^3 y \qquad \text{subject to} \quad y(1) = 1.$$ [3 marks]

Question 2.

The Laplace Transform of $f(t)$ is defined according to

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt.$$

(a) Use the above definition to find the Laplace Transform of the function e^{-at}. Hence write down $\mathcal{L}[1]$. [2 marks]

(b) Use the above definition of the Laplace Transform to find the transform of both $f'(t)$ and $f''(t)$ in terms of $F(s)$, $f(0)$ and $f'(0)$. [3 marks]

(c) Hence solve the equation,

$$y'' + 2y' = -2e^{-t},$$

subject to the initial conditions, $y(0) = 0$ and $y'(0) = 0$. [5 marks]

Question 3.

(a) Find the determinant of the matrix,

$$A = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 1 & 3 \\ 2 & 1 & -1 & 1 \\ 2 & 4 & -2 & 2 \end{pmatrix}$$

[4 marks]

(b) Use Gaussian Elimination to solve the following system of simultaneous equations:

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 1 & -6 \\ -2 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}.$$

[6 marks]

[Note: Full marks will only be awarded for a correct answer which has used the Gaussian Elimination method in its precise form, i.e. where the matrix has been reduced to upper triangular form.]

Question 4.

(a) The function, $f(t)$, has a period equal to 2π and it is given by $f(t) = t^2$ within the range, $-\pi < t < \pi$. Sketch the function in the range $-3\pi < t < 3\pi$ and find its Fourier series representation. (The definitions of the Fourier coefficients are given below.) [6 marks]

(b) Hence find the particular integral of the ordinary differential equation,

$$\frac{d^2y}{dt^2} + \pi^2 y = f(t).$$

Clearly, $y(t)$ will be continuous at $t = \pi$, but how many of its derivatives will also be continuous at that point? [4 marks]

You may use the following definitions of the Fourier Coefficients,

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \cos nt + B_n \sin nt\Big],$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, \cos nt\, dt \qquad n = 0, 1, \cdots, \infty,$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, \sin nt\, dt \qquad n = 1, 2, \cdots, \infty.$$

Question 5.

(a) The following data have been obtained from an experiment.

x_i	0	1	2	3	4
y_i	1.10	0.62	0.41	-0.10	-0.35

It is proposed to fit the curve, $y = a + bx$, to this data using the method of least squares. Develop the theory, and find the line of best fit. [7 marks]

(b) With a different group of n data points (not provided here) it is proposed to fit $y = a\cos x + b\sin x$. Derive the theory which one would need to apply to determine a and b. [3 marks]

Question 6.

(a) It is intended to use an *ad hoc* iteration scheme to solve the following cubic equation, $x^3 - 7x + 6 = 0$. There are two possible ways that this may be done; write down both of these iteration schemes, and then find four iterations for each scheme beginning with $x_0 = 1.1$. [5 marks]

(b) For each scheme, analyze the approach to the root $x = 1$ by setting $x_n = 1 + \epsilon_n$ in the respective formulae, where ϵ_n is very small, and by evaluating x_{n+1}. Comment on their convergence properties. [5 marks]

Question 7.

Solve the ordinary differential equation,

$$\frac{d^2y}{dt^2} + y = 2\cos t,$$

subject to the initial conditions

$$y = 0, \quad \frac{dy}{dt} = 1 \quad \text{at} \quad t = 0.$$

[10 marks]

Question 8.

The Laplace Transform is defined in Question 2.

For the present question you may assume the result, $\mathcal{L}[\cos at] = s/(s^2+a^2)$. The convolution theorem states that,

$$\mathcal{L}[f * g] = F(s)G(s) \qquad \text{where} \quad \mathcal{L}[f] = F(s) \;\text{ and }\; \mathcal{L}[g] = G(s),$$

and where the convolution of f and g is defined as

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

(a) Given that $\mathcal{L}[f(t)] = F(s)$ and that $b > 0$, find the Laplace Transform of the function $H(t-b)f(t-b)$ in terms of $F(s)$, where H denotes the unit step function. Hence find the inverse transform of the function,

$$e^{-bs}\frac{s}{s^2+a^2}.$$

[4 marks]

(b) Find the Laplace Transform of the unit impulse at $t = b$, namely, $\delta(t-b)$. [2 marks]

(c) Given the result of part (b), use the convolution theorem to find, by a different approach, the inverse transform of the function given in part (a). [4 marks]

Question 9.

(a) Find all the eigenvalues of the matrix,

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

and determine their corresponding eigenvectors. [6 marks]

(b) Use the result of part (a) to write down the general solution of the following system of ordinary differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The initial condition is $(x, y, z) = (2, 3, 1)$ at $t = 0$; what is the final solution? [4 marks]

Question 10.

In this question we shall be solving the following inhomogeneous system of ordinary differential equations in two different ways.

$$\frac{dy}{dt} - z = 0, \qquad (10.1)$$

$$\frac{dz}{dt} + 2z + y = e^{-t}. \qquad (10.2)$$

subject to $y(0) = z(0) = 0$.

(a) The definition of the Laplace Transform is given in Q2. You may assume that $\mathcal{L}[y'] = sY(s) - y(0)$ when $\mathcal{L}[y] = Y(s)$. Also use the notation that $\mathcal{L}[z] = Z(s)$.

Find $\mathcal{L}[t^2]$ and prove the s-shift theorem, namely that $\mathcal{L}[e^{-at}y(t)] = Y(s+a)$. [3 marks]

(b) Take the Laplace Transform of each of the above equations for y and z, then eliminate Z between the resulting algebraic equations for Y and Z, and finally use part (a) to determine $y(t)$. [3 marks]

(c) Eliminate z from between Eqs. (10.1) and (10.2) and thereby obtain a second order equation for y. Then use standard Complementary Function and Particular Integral techniques to solve for y subject to the given initial conditions. Give an expression for z. [4 marks]