UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2018
Unit Title: Mathematics 2		Unit Code: ME10305
Year: 2017/18	Question Number: 1	Page 1 of 1

Part		Mark
(a)	$\left.\begin{array}{l} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{array}\right\} \Rightarrow \begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 1 - y_1 y_3 - 2y_1 y_2 \end{array}$ $\quad x=0 \Rightarrow y_1 = y_2 = 0$, $\quad x = 1 \Rightarrow y_1 = 1$	2
	3rd order, nonlinear, BVP	2
(b)	First rearrange into the correct 1° linear form:	
	$y' - \frac{y}{t} = -t^2$	
	I.F. is $e^{\int -\frac{1}{t}dt} = e^{-\ln t} = 1/t$	
	Hence $\frac{y'}{t} - y/t^2 = -t \Rightarrow (y/t)' = -t$	
	$\Rightarrow y/t = c - t^2/2 \Rightarrow y = ct - t^3/2$	
	I.C. $y(1) = 1 \Rightarrow c = 3/2$. Hence $\boxed{y = \frac{3}{2}t - t^3/2}$	3
(c)	We have $t\frac{dy}{dt} = y(1-t^3)$	
	$\Rightarrow \frac{dy}{y} = \left(\frac{1}{t} - t^2\right)dt$	
	$\Rightarrow \ln\lvert y\rvert = \ln\lvert t\rvert - t^3/3 + c$	
	$\Rightarrow y = Ate^{-t^3/3}$	
	I.C. $y(1) = 1 \Rightarrow A = e^{1/3}$	
	$\Rightarrow \boxed{y = te^{(1-t^3)/3}}$	3
	Total	10

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(a)	$\mathcal{L}[e^{-at}] = \int_0^\infty e^{-at} e^{-st}\,dt = \boxed{\frac{1}{s+a}}$ $\mathcal{L}[1] = \boxed{\frac{1}{s}}$ using $a=0$	2
(b)	$\mathcal{L}[f'] = \int_0^\infty f' e^{-st}\,dt = [f][e^{-st}]_0^\infty - \int_0^\infty [f][-se^{-st}]\,dt$ $= sF(s) - f(0)$ $\mathcal{L}[f''] = \int_0^\infty f'' e^{-st}\,dt = [f'][e^{-st}]_0^\infty - [f][-se^{-st}]_0^\infty + \int_0^\infty [f][s^2 e^{-st}]\,dt$ $= s^2 F(s) - sf(0) - f'(0)$	3
(c)	$y'' + 2y' = -2e^{-t}$ LT $\Rightarrow$ $(s^2+2s)Y = \frac{-2}{s+1}$ $\Rightarrow Y = \frac{-2}{s(s+1)(s+2)}$ $= -\frac{1}{s} + \frac{2}{s+1} - \frac{1}{s+2}$ using Partial Fractions $\Rightarrow \boxed{y = -1 + 2e^{-t} - e^{-2t}.}$	5
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Part		Mark

(a)

$$\begin{vmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 1 & 3 \\ 2 & 1 & -1 & 1 \\ 2 & 4 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 3 & 1 \\ 0 & -5 & 1 & -1 \\ 0 & -2 & 0 & 0 \end{vmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix} = \begin{vmatrix} -6 & 3 & 1 \\ -5 & 1 & -1 \\ -2 & 0 & 0 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = 8$$

4

(b) Augmented matrix notation:

$$\left(\begin{array}{ccc:c} -1 & 2 & 2 & 3 \\ 2 & 1 & -6 & 2 \\ -2 & 4 & 5 & 7 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc:c} -1 & 2 & 2 & 3 \\ 0 & 5 & -2 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - 2R_1 \end{matrix}$$

Unusually, now in upper triangular form now.

$$\Rightarrow z = 1, \quad y = 2, \quad x = 1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

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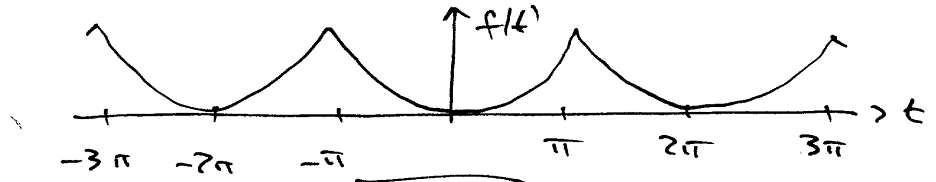
Part		Mark
(a)		1
	Even function $\Longrightarrow \boxed{B_n = 0}$	1
	$A_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2}{3}\pi^2$	
	$A_n = \frac{2}{\pi}\int_0^{\pi} t^2 \cos nt\,dt$	
	$= \frac{2}{\pi}\Big[(t^2)\big(\cancel{\tfrac{\sin nt}{n}}\big) - (2t)\big(-\tfrac{\cos nt}{n^2}\big) + (2)\big(-\cancel{\tfrac{\sin nt}{n^3}}\big)\Big]_0^{\pi}$	
	$= \frac{2}{\pi}\Big[\frac{2\pi \cos \pi n}{n^2}\Big] = \frac{4}{n^2}(-1)^n.$	
	$\Longrightarrow \boxed{f(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nt}$	4
(b)	$\boxed{y_{PI} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2(\pi^2 - n^2)} \cos nt.}$	3
	The denominator $(n^2(\pi^2 - n^2))$ increases as n^4	
	& hence $\boxed{y, y', y'' \text{ are continuous at } t = \pi.}$	1
	Total	10

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Part		Mark
(a)	$y = a + bx \Rightarrow S = \sum (bx_i + a - y_i)^2$	
	$0 = \frac{\partial S}{\partial a} = 2\sum (bx_i - y_i + a)$	
	$0 = \frac{\partial S}{\partial b} = 2\sum x_i (bx_i + a - y_i)$	
	$\Rightarrow \begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$	3
	Using the numbers given:	
	$\begin{pmatrix} 5 & 10 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1.68 \\ -0.27 \end{pmatrix}$	
	$\Rightarrow \boxed{\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1.062 \\ -0.309 \end{pmatrix}}$ $\quad y = 1.062 - 0.309x$	4
(b)	If $y = a\cos x + b\sin x$ then	
	$\begin{pmatrix} \sum \cos^2 x_i & \sum \cos x_i \sin x_i \\ \sum \cos x_i \sin x_i & \sum \sin^2 x_i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i \cos x_i \\ \sum y_i \sin x_i \end{pmatrix}$	3
	Total	10

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Part		Mark
(a)	The two schemes are $A: \quad x_{n+1} = (7x_n - 6)^{1/3}$ $B: \quad x_{n+1} = (x_n^3 + 6)/7.$	2
	n: 0, 1, 2, 3, 4 A x_n: 1.1, 1.193 483, 1.330 329, 1.490 653, 1.642 923 B x_n: 1.1, 1.047 286, 1.021 239, 1.009 297, 1.004 027	3
(b)	$A: \quad x_{n+1} = [7(1+\varepsilon) - 6]^{1/3}$ $= [1 + 7\varepsilon]^{1/3}$ $\simeq 1 + \frac{7}{3}\varepsilon$ — diverges linearly	3
	$B: \quad x_{n+1} = \dfrac{(1 + 3\varepsilon + \dots) + 6}{7}$ $\simeq 1 + \frac{3}{7}\varepsilon \dots$ — converges linearly.	2
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Part		Mark

$y'' + y = 2\cos t$

CF $\lambda^2 + 1 \Rightarrow \lambda = \pm j$

$\Rightarrow \boxed{y_{CF} = A\cos t + B\sin t}$. 3

PI Also $\lambda = \pm j$ so

$y_{PI} = Ct\cos t + Dt\sin t$.

Substitute into ODE to get

$-2C\sin t + 2D\cos t = 2\cos t$

$\Rightarrow C = 0,\ D = 1$.

$\therefore \boxed{y_{PI} = t\sin t}$. 5

Hence the general solution is

$y = A\cos t + B\sin t + t\sin t$

$y(0) = 0 \Rightarrow A = 0$

$y'(0) = 1 \Rightarrow B = 1$

$\Rightarrow \boxed{y = (t+1)\sin t}$. 2

Total	10

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Part		Mark
(a)	$\mathcal{L}[H(t-b)f(t-b)] = \int_0^\infty H(t-b)f(t-b)\underset{\uparrow\, e^{-st}}{}dt = \int_b^\infty f(t-b)e^{-st}\,dt$	
	$\overset{(\tau=t-b)}{=\!=} \int_0^\infty f(\tau)e^{-s\tau}e^{-sb}\,d\tau = e^{-sb}F(s)$	2
	If $f(t)=\cos at$ then $\mathcal{L}^{-1}\left[e^{-sb}\frac{s}{s^2+a^2}\right]$	
	$= H(t-b)\cos a(t-b)$.	2
(b)	$\mathcal{L}[\delta(t-b)] = \int_0^\infty \delta(t-b)e^{-st}\,dt = e^{-sb}$	
	using properties of $\delta(t)$.	2
(c)	$f*g = \mathcal{L}^{-1}[F(s)G(s)]$	
	If $F=e^{-sb}$ & $G=\frac{s}{s^2+a^2}$	
	then $f=\delta(t-b)$ & $g=\cos at$.	
	Hence $\mathcal{L}^{-1}\left[e^{-sb}\frac{s}{s^2+a^2}\right] = \delta(t-b)*\cos at$	
	$= \int_0^t \delta(\tau-b)\cos a(t-\tau)\,d\tau$	
	$= \begin{cases} 0 & t\leqslant b \\ \cos a(t-b) & t>b \end{cases}$	
	$= H(t-b)\cos a(t-b)$.	4
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Part		Mark
(a)	$0 = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(3-\lambda)(1-\lambda)-1\right]-2(1-\lambda)$	
	$= (1-\lambda)\left[\lambda^2-4\lambda\right] = \lambda(1-\lambda)(\lambda-4)$	
	$\Rightarrow \lambda = 0, 1, 4$ are the eigenvalues.	1
	$\lambda=0 \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$	2
	$\lambda=1 \Rightarrow \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	2
	$\lambda=4 \Rightarrow \begin{pmatrix} -3 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = C\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$	2
(b)	General solution is	
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + B\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{t} + C\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}e^{4t}$	2
	If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ at $t=0$ then $A=B=0$, $C=1$.	
	Hence $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}e^{4t}$	1
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Part		Mark
(a)	$\mathcal{L}[t^2] = \int_0^\infty t^2 e^{-st}\,dt = [t^2]\left[\frac{e^{-st}}{-s}\right]_0^\infty - [2t]\left[\frac{e^{-st}}{s^2}\right]_0^\infty + [2]\left[\frac{e^{-st}}{-s^3}\right]_0^\infty$	2
	$= 2/s^3$	
	$\mathcal{L}[e^{-at}y(t)] = \int_0^\infty y(t)\, e^{-(s+a)t}\,dt = Y(s+a).$	1
(b)	We get $\quad sY - Z = 0$	
	$sZ + 2Z + Y = \frac{1}{s+1}$	
	$\Rightarrow (s^2+2s+1)Y = \frac{1}{s+1} \quad$ on eliminating Z.	
	$\Rightarrow Y = \frac{1}{(s+1)^3}$	
	Use part (a) $\quad \mathcal{L}[t^2] = 2/s^3 \Rightarrow \mathcal{L}[t^2 e^{-at}] = \frac{2}{(s+a)^3}$	
	Hence $\boxed{y = \frac{1}{2}t^2 e^{-t}}$ $\quad$ ($a=1$ & divide by 2).	3
(c)	If $z = y'$ then $\quad \underbrace{y'' + 2y' + y}_{\lambda=-1,-1} = \underbrace{e^{-t}}_{\lambda=-1} \quad$ ($e^{\lambda t}$ trick)	
	$\therefore\ y = \underbrace{(A+Bt)e^{-t}}_{CF} + \underbrace{Ct^2e^{-t}}_{PI}.$	
	After substitute we find $C = 1/2$.	
	Using ICs $y(0)=0 \Rightarrow A=0$, $y'(0)=0 \Rightarrow B=0$	
	we have $y = \frac{1}{2}t^2 e^{-t}$.	
	So $z = \frac{1}{2}[2t - t^2]e^{-t}$.	4

Total	10