Each question carries 10 marks.

The examination consists of TEN questions.
All questions should be attempted.

The marks shown against each **part** *of a question are for guidance only.*

Candidates are permitted to use University-approved calculators.

Question 1.

(a) Write the following ordinary differential equation and its boundary conditions in first order form:

$$\frac{d^4y}{dx^4} + y\frac{dy}{dx} + 2y = 1,$$

where the boundary conditions are that,

$$y = \frac{dy}{dx} = 0 \quad \text{at both} \quad x = 0 \quad \text{and} \quad x = 1.$$

[2 marks]

Classify this equation and its boundary conditions with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem (giving reasons). [2 marks]

(b) Use the method of Integrating Factors to solve the equation,

$$\frac{dy}{dt} + \frac{y}{t+1} = 1,$$

subject to the initial condition, $y(0) = 1/2$. [3 marks]

(c) Use the technique of separation of variables to solve the equation,

$$(t^2 + 1)\frac{dv}{dt} = -4vt,$$

subject to $v = 1$ when $t = 0$. [3 marks]

Question 2.

The Laplace Transform of $f(t)$ is defined according to

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\, dt.$$

(a) Use the above definition to find the Laplace Transform of the function $\cos\omega t$. [3 marks]

(b) Use the above definition of the Laplace Transform to find the transform of $f''(t)$ in terms of $F(s)$, $f(0)$ and $f'(0)$. [3 marks]

(c) Hence solve the equation,

$$y'' + y = 3\cos 2t,$$

subject to the initial conditions, $y(0) = 0$ and $y'(0) = 0$. [4 marks]

Question 3.

(a) The function, $f(t)$, has a period equal to 2π and it is given by $f(t) = t$ within the range, $-\pi < t < \pi$. Find its Fourier series representation. (The definitions of the Fourier coefficients are given below.) [7 marks]

(b) Hence find the particular integral of the ordinary differential equation,

$$\frac{d^2y}{dt^2} + \pi^2 y = f(t).$$

[3 marks]

You may use the following definitions of the Fourier Coefficients,

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \cos nt + B_n \sin nt\Big],$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, \cos nt \, dt \qquad n = 0, 1, \cdots, \infty,$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, \sin nt \, dt \qquad n = 1, 2, \cdots, \infty.$$

Question 4.

(a) Find the determinant of the matrix,

$$A = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

[4 marks]

(b) Use Gaussian Elimination to solve the following system of simultaneous equations:

$$\begin{pmatrix} -1 & 2 & 1 \\ -2 & 5 & 3 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}.$$

[6 marks]

[Note: Full marks will only be awarded for a correct answer which has used the Gaussian Elimination method in its precise form, i.e. where the matrix has been reduced to upper triangular form.]

Question 5.

The following data have been obtained from an experiment.

x_i	1	1.25	1.5	1.75	2
y_i	1.01	1.22	1.46	1.63	1.83

It is proposed to fit the curve, $y = a + b\sqrt{x}$, to this data using the method of least squares. Develop the theory, and find the line of best fit. [10 marks]

Question 6.

(a) In two separate computations use $N = 4$ and $N = 8$ equally-spaced intervals to approximate the integral,

$$\int_0^1 \frac{1}{1+x}\,dx$$

using the trapezium rule. Solve the integral analytically and find the exact error for each value of N. [4 marks]

The trapezium rule has second order accuracy. Use Richardson's extrapolation formula to obtain a more accurate numerical value of the integral. What is the error for this extrapolate? [2 marks]

(b) A new numerical integration method has been devised to perform numerical integrations, but its order of accuracy is unknown. Therefore the method has been applied for N intervals, $2N$ intervals and $4N$ intervals. Derive the formula for the ratio test to estimate the order of accuracy. [2 marks]

(c) The following data has been obtained, where I_N denotes the numerical approximation to the exact integral, I, using N intervals:

N	I_N
100	2.704814
200	2.711517
400	2.714892

Using the ratio test formula devised in part (b), determine the order of accuracy of the method which produced this data, and then use Richardson's extrapolation on I_{200} and I_{400} to determine an even more accurate estimate of the integral. [2 marks]

Question 7.

The matrix, A, is defined as follows:

$$A = \begin{pmatrix} 10 & 4 \\ 9 & 10 \end{pmatrix}.$$

(a) Find all the eigenvalues and eigenvectors of A. [4 marks]

(b) Using the results of part (a), determine the general solution of the following system of ordinary differential equations,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 9 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

[2 marks]

(c) Hence find that solution which satisfies the initial conditions,

$$x(0) = 1, \qquad y(0) = 0.$$

[2 marks]

(d) Using the information found in parts (a) and (b), find the general solution of the following system of ordinary differential equations,

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 9 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

[2 marks]

Question 8.

Use the substitution $y(t) = z(t)e^{-2t}$ to transform the ordinary differential equation,

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 2e^{-2t},$$

into an ordinary differential equation for $z(t)$. Solve this equation for z and hence write down the general solution for $y(t)$. What is the specific solution which satisfies the initial conditions, $y(0) = 1$ and $y'(0) = 0$? [10 marks]

Question 9.

The Laplace Transform of $f(t)$ is defined in Question 2.

(a) Use the definition of the Laplace Transform to find both $\mathcal{L}[e^{-at}]$ and $\mathcal{L}[t]$. [2 marks]

(b) Use the definition of the Laplace Transform to prove the s-shift theorem

$$\mathcal{L}[f(t)e^{-at}] = F(s+a)$$

where $\mathcal{L}[f(t)] = F(s)$. [2 marks]

(c) Use the s-shift theorem to determine the inverse Laplace Transform of

$$\frac{1}{s^2+4s+4}.$$

[3 marks]

(d) The convolution theorem is $\mathcal{L}[f * g] = F(s)G(s)$, where

$$\mathcal{L}[f(t)] = F(s), \quad \mathcal{L}[g(t)] = G(s) \quad \text{and} \quad f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Use the convolution theorem as an alternative way to find the inverse Laplace Transform of

$$\frac{1}{s^2+4s+4}.$$

[3 marks]

Question 10.

(a) Use the definition of the Newton-Raphson iteration scheme for $f(x) = x^3 - 8 = 0$, to show that the formula given below may be used to find the cube root of 8:

$$x_{n+1} = \frac{2x_n^3 + 8}{3x_n^2}.$$

Verify that this formula does find the cube root of 8 by evaluating four successive iterates beginning with $x_1 = 3$. [5 marks]

(b) Analyse the approach to the limit $x_\infty = 2$ by setting $x_n = 2 + \epsilon_n$ in the above formula, where ϵ_n is very small, and by evaluating x_{n+1}. [5 marks]