UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2017
Unit Title: Mathematics 2		Unit Code: ME10305
Year: 2016/17	Question Number: 1	Page 1 of 1

Part		Mark
(a)	Let $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $y_4 = y'''$ $\Rightarrow$ $y_1' = y_2$, $y_2' = y_3$, $y_3' = y_4$, $y_4' = 1 - 2y_1 - y_1 y_2$. BCs: $y_1 = y_2 = 0$ at $x = 0, 1$	2
	4^{th} order, nonlinear, BVP	2
(b)	Eq $\rightarrow$ $y' + \frac{1}{1+t}y = 1$ Int. Fact $= e^{\int \frac{dt}{1+t}} = e^{\ln\|1+t\|} = 1+t$ $\Rightarrow (1+t)y' + y = 1+t$ $\Rightarrow (1+t)y = t + t^2/2 + c$ $y(0) = 1/2 \Rightarrow c = 1/2 \Rightarrow (1+t)y = \frac{1}{2}(1+2t+t^2) = 1/2(1+t)^2$ $\Rightarrow y = \frac{1}{2}(1+t)$	3
(c)	Eq $\rightarrow$ $\frac{dv}{v} = -\frac{4t}{t^2+1}dt = -2 \times \frac{2t}{t^2+1}dt$ $\Rightarrow \ln v = -2\ln(t^2+1) + c = -2\ln A(t^2+1)$ $\Rightarrow v = B(t^2+1)^{-2}$ $t = 0 \Rightarrow v = 1 \Rightarrow B = 1 \Rightarrow v = \frac{1}{(t^2+1)^2}$	3
	Total	10

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(a)	Could use integration by parts or $\mathcal{L}[\cos\omega t] = \mathrm{Re}\,\mathcal{L}[e^{j\omega t}] = \mathrm{Re}\int_0^\infty e^{j\omega t}e^{-st}\,dt.$ $= \mathrm{Re}\left(\frac{1}{s-j\omega}\right) = \mathrm{Re}\left(\frac{s+j\omega}{s^2+\omega^2}\right) = \frac{s}{s^2+\omega^2}$	3
(b)	$\mathcal{L}[f''] = \int_0^\infty f'' e^{-st}\,dt$ $= [f'][e^{-st}]_0^\infty - [f][-se^{-st}]_0^\infty + \int_0^\infty [f][s^2e^{-st}]\,dt$ $= -f'(0) - sf(0) + s^2F(s)$	3
(c)	$y'' + y = 3\cos 2t$ $\longrightarrow \quad s^2Y + Y = \frac{3s}{s^2+4}$ $\Rightarrow Y = \frac{3s}{(s^2+1)(s^2+4)} = \frac{s}{s^2+1} - \frac{s}{s^2+4}$ using Partial Fractions. Hence part (a) gives $y = \cos t - \cos 2t$	4
	Total	10

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Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2017
Unit Title: Mathematics 2		Unit Code: ME10305
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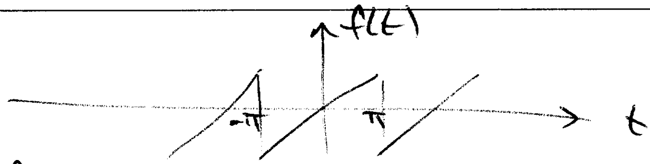
Part		Mark
(a)	 The function is odd $\Rightarrow A_n = 0$. So $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin nt \, dt$ $= \frac{1}{\pi}\Big[[t][-\frac{\cos nt}{n}] - [1][-\frac{\sin nt}{n^2}] \Big]_{-\pi}^{\pi}$ $= -\frac{1}{n\pi}\big[t \cos nt \big]_{-\pi}^{\pi} = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$ $\Rightarrow f(t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nt$	7
(b)	$y'' + \pi^2 y = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nt$ Let $y = \sum_{n=1}^{\infty} C_n \sin nt$ $\Rightarrow \sum_{n=1}^{\infty} C_n(\pi^2 - n^2) \sin nt = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nt$ $\Rightarrow C_n = \frac{2(-1)^{n+1}}{n(\pi^2 - n^2)} = \frac{2(-1)^n}{n(n^2 - \pi^2)}$ $\therefore y = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n(n^2 - \pi^2)} \sin nt$	3
	Total	10

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Outline Solution to Examination Question

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Part		Mark
(a)	$\begin{vmatrix} 1 & 3 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & \underset{\uparrow\ C_3+C_4}{4} & 3 \end{vmatrix} = -4\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$ $= -4\begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} R_1 - R_3 \quad = -(-4)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ $= 4$	4
(b)	$\left(\begin{array}{ccc\|c} -1 & 2 & 1 & 2 \\ -2 & 5 & 3 & 7 \\ 1 & -3 & 0 & -3 \end{array}\right)$ $\left(\begin{array}{ccc\|c} -1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & -1 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$ $\left(\begin{array}{ccc\|c} -1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right) \begin{array}{l} \\ \\ R_3+R_2 \end{array}$ Back-substitution $\Rightarrow z=1,\ y=2,\ x=3$	6
		Total 10

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Outline Solution to Examination Question

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Part		Mark

Let $S = \sum_i \left(y_i - a - b\sqrt{x_i}\right)^2$

$$0 = \frac{\partial S}{\partial a} = -2\sum\left(y_i - a - b\sqrt{x_i}\right) \Rightarrow aN + b\sum_i \sqrt{x_i} = \sum y_i$$

$$0 = \frac{\partial S}{\partial b} = -2\sum \sqrt{x_i}\left(y_i - a - b\sqrt{x_i}\right) \Rightarrow a\sum\sqrt{x_i} + b\sum x_i = \sum y_i\sqrt{x_i}$$

or
$$\begin{pmatrix} N & \sum_i \sqrt{x_i} \\ \sum_i \sqrt{x_i} & \sum_i x_i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_i y_i \\ \sum_i \sqrt{x_i}\, y_i \end{pmatrix}$$ 5

Hence
$$\begin{pmatrix} 5 & 6.079868 \\ 6.079866 & 7.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 7.15 \\ 8.906427 \end{pmatrix}$$ 3

$$\Rightarrow a = -0.980768$$

$$b = 1.982569$$ 2

Total 10

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Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2017
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Part		Mark
(a)	$I_N = h[\frac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{N-1} + \frac{1}{2}f_N]$	
	From the data $I_4 = 0.697024$ $I_8 = 0.694122$	
	Exact: $I = \int_0^1 \frac{dx}{1+x} = [\ln\lvert x+1\rvert]_0^1 = \ln 2 = 0.693147$	
	$\text{Error}_4 = I - I_4 = -0.003877$	
	$\text{Error}_8 = I - I_8 = -0.000975$	4
	$\bar{I}_8 = \frac{4I_8 - I_4}{3} = 0.693155$	
	$\Rightarrow \text{Error}_{\text{EXTRAP}} = -0.000008$.	2
(b)	$\left.\begin{aligned} I_N &= I + Ah^\alpha \\ I_{2N} &= I + A(h/2)^\alpha \\ I_{4N} &= I + A(h/4)^\alpha \end{aligned}\right\} \Rightarrow \frac{I_N - I_{2N}}{I_{2N} - I_{4N}} \simeq 2^\alpha$.	2
(c)	Using the given data	
	$\frac{I_N - I_{2N}}{I_{2N} - I_{4N}} = 1.9861$	
	Therefore $\alpha = 1$ is very likely.	1
	$\therefore \bar{I}_{400} = 2I_{400} - I_{200}$	
	$= 2.718267$	1
		Total 10

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Part		Mark
(a)	$0 = \begin{vmatrix} 10-\lambda & 4 \\ 9 & 10-\lambda \end{vmatrix} = (\lambda-10)^2 - 36 \Rightarrow \lambda = 10 \pm 6 = 4, 16.$ For $\lambda = 4$ $\begin{pmatrix} 6 & 4 \\ 9 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ For $\lambda = 16$ $\begin{pmatrix} -6 & 4 \\ 9 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = B\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$	4
(b)	If $\begin{pmatrix} x \\ y \end{pmatrix} = e^{\lambda t}\begin{pmatrix} X \\ Y \end{pmatrix}$ we get $\begin{pmatrix} 10-\lambda & 4 \\ 9 & 10-\lambda \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Hence $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} 2 \\ -3 \end{pmatrix}e^{4t} + B\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{16t}.$	2
(c)	If $x(0) = 1$ & $y(0) = 0$ then. $A = \frac{1}{4}$, $B = 1/4$. $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ -3 \end{pmatrix}e^{4t} + \frac{1}{4}\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{16t}.$	2
(d)	Same as part (b) except that $\lambda^2 = 4, 16$ $\therefore \lambda = \pm 2, \pm 4$ Solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \left[Ae^{2t} + Be^{-2t}\right]\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \left[Ce^{4t} + De^{-4t}\right]\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	2
	Total	10

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Part		Mark
	If $y = z e^{-2t}$ then $y' = e^{-2t}(z' - 2z)$	
	$y'' = e^{-2t}(z'' - 4z' + 4z)$	
	$\therefore$ $y'' + 4y' + 4y = 2e^{-2t}$ becomes	
	$\left[(z'' - 4z' + 4z) + 4(z' - 2z) + 4z\right]e^{-2t} = e^{-2t}$	
	$\Rightarrow z'' = 2$	4
	$\Rightarrow z = \frac{t^2}{\cancel{4}} + At + B$	
	$\Rightarrow y = \left[\frac{t^2}{\cancel{4}} + At + B\right]e^{-2t}$	1
	ICs are $y(0) = 1 \Rightarrow B = 1$.	1
	$y'(0) = 0$ — but $y' = -2\left[\frac{t^2}{\cancel{4}} + At + B\right]e^{-2t}$	
	$+ \left[2t + A\right]e^{-2t}$	
	$\Rightarrow y'(0) = -2B + A$	
	$\Rightarrow A = 2B = 2$.	3
	Hence $y = \left[\frac{t^2}{\cancel{4}} + 2t + t\right]e^{-2t}$	
	$= (t+1)^2 e^{-2t}$	1
	Total	10

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Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date May 2017
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Part		Mark
(a)	$\mathcal{L}[e^{-at}] = \int_0^\infty e^{-at}e^{-st}\,dt = \int_0^\infty e^{-(s+a)t}\,dt = \frac{1}{s+a}$	1
	$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt = [t]\left[\frac{e^{-st}}{-s}\right]_0^\infty - [1]\left[\frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}$	1
(b)	$\mathcal{L}[f(t)e^{-at}] = \int_0^\infty f(t)e^{-at}e^{-st}\,dt$ $= F(s+a)$	2
(c)	If $F(s) = \frac{1}{s^2}$ then $F(s+2) = \frac{1}{s^2+4s+4}$ So $f(t) = t \Longrightarrow \mathcal{L}^{-1}\left[\frac{1}{s^2+4s+4}\right] = te^{-2t}$.	3
(d)	As $\frac{1}{s^2+4s+4} = \frac{1}{s+2} \times \frac{1}{s+2}$, let $F = G = \frac{1}{s+2}$. $\Rightarrow f(t) = g(t) = e^{-2t}$ from part (a). $\therefore \mathcal{L}^{-1}\left[\frac{1}{s^2+4s+4}\right] = e^{-2t} * e^{-2t}$ $= \int_0^t e^{-2\tau}e^{-2(t-\tau)}\,d\tau$ $= \left(e^{-2t}\right)\int_0^t 1\,d\tau = te^{-2t}$.	5
	Total	10

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Part		Mark
(a)	$x_{n+1} = x_n - f(x_n)/f'(x_n)$ $\Rightarrow x_{n+1} = x_n - \dfrac{(x_n^3 - 8)}{3x_n^2} = \dfrac{2x_n^3 + 8}{3x_n^2}$	2
	Using this we get $3 \longrightarrow 2.296296$ $\longrightarrow 2.036587$ $\longrightarrow 2.000653$ $\longrightarrow 2.0000002$	3
(b)	Let $x_n = 2 + \varepsilon$ in $x_{n+1} = \dfrac{2x_n^3 + 8}{3x_n^2}$ $\Rightarrow x_{n+1} = \dfrac{2[8 + 12\varepsilon + 6\varepsilon^2 + \varepsilon^3] + 8}{3[4 + 4\varepsilon + \varepsilon^2]}$ $= \dfrac{24 + 24\varepsilon + 12\varepsilon^2 + 2\varepsilon^3}{12 + 12\varepsilon + 3\varepsilon^2}$ $= \dfrac{2(12 + 12\varepsilon + 3\varepsilon^2) + 6\varepsilon^2 + 2\varepsilon^3}{12 + 12\varepsilon + 3\varepsilon^2}$ $\simeq 2 + \varepsilon^2/2$ Hence quadratic convergence.	5
		Total 10