Each question carries 10 marks.

The examination consists of TEN questions.
All questions should be attempted.

The marks shown against each **part** *of a question are for guidance only.*

Candidates are permitted to use University-approved calculators.

Question 1.

(a) Write the following system of ordinary differential equations and their boundary conditions in first order form:

$$\frac{d^3 f}{dx^3} + f\frac{d^2 f}{dx^2} - g = 0, \qquad \frac{d^2 g}{dx^2} + f\frac{dg}{dx} = 0,$$

subject to,

$$f = \frac{df}{dx} = 0 \quad \text{and} \quad g = 1 \quad \text{at} \quad x = 0,$$

and

$$\frac{df}{dx} \to 1, \quad g \to 0 \quad \text{as} \quad x \to \infty.$$

[2 marks]

Classify this system with regard to its order, whether it is linear or nonlinear, and whether it is an initial value problem or a boundary value problem (giving reasons). [2 marks]

(b) Use the technique of separation of variables to solve the equation,

$$(t^2 + 1)\frac{dy}{dt} = yt,$$

subject to $y(0) = 1$. [3 marks]

(c) Find the solution of the equation,

$$\frac{dy}{dt} + \frac{3y}{t} = \frac{2}{t^2},$$

subject to $y = 0$ when $t = 1$, using the integrating factor method. [3 marks]

Question 2.

The Laplace Transform of $f(t)$ is defined according to

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt.$$

(a) Use the above definition of the Laplace Transform to find the Laplace Transform of the function $\sin \omega t$. [3 marks]

(b) Use the above definition of the Laplace Transform to find $\mathcal{L}[f''(t)]$ in terms of $F(s)$, $f(0)$ and $f'(0)$. [2 marks]

(c) Hence solve the equation,

$$y'' + 4y = 3\sin t,$$

subject to the initial conditions, $y(0) = 0$ and $y'(0) = 3$. [5 marks]

Question 3.

Use the method of Gaussian Elimination to solve the following system of simultaneous equations:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 8 \\ 13 \end{pmatrix}.$$

[10 marks]

[Note: Full marks will only be awarded for a correct answer which has used the Gaussian Elimination method in its precise form, i.e. where the matrix has been reduced to upper triangular form.]

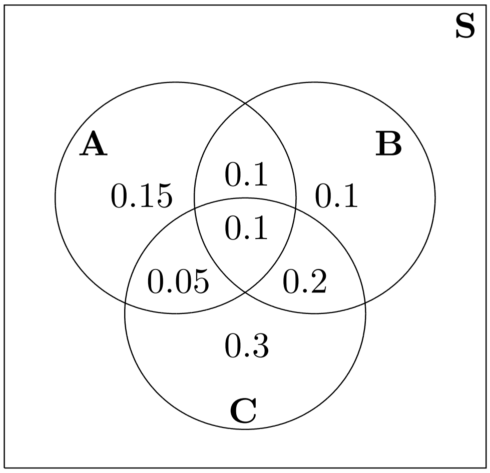
Question 4.

(a) You have in your possession two eight-sided octahedral fair dice each with faces numbered from 1 to 8, inclusive. It is assumed that each number is equally likely.

Both dice are thrown, and the face values are added to give a total score of between 2 and 16, inclusive. Determine the probabilities for each of the possible total scores. [3 marks]

What are the probabilities of obtaining, (i) an even score, (ii) a score which is strictly less than 5, and (iii) a score which is a square number? [3 marks]

(b) The following Venn diagram shows the probabilities associated with the various subsets A, B and C of sample space. Find both $P(B|A)$ and $P(C|A)$. Are the events A and B independent? Are the events A and C independent? [4 marks]



Question 5.

The following data have been obtained from an experiment.

x_i	0	1	2	3	4
y_i	-0.02	1.21	3.98	9.17	15.71

It is proposed to fit the quadratic curve, $y = ax^2 + b$, to this data using the method of least squares. Develop the theory, and then find the line of best fit. [7 marks]

Suppose now that the fitted curve must pass through the origin. What is now the curve of best fit? [3 marks]

Question 6.

The Laplace Transform is defined in Question 2.

For the present question you may assume the result, $\mathcal{L}[\cos at] = s/(s^2+a^2)$. The convolution theorem states that,

$$\mathcal{L}[f * g] = F(s)G(s),$$

where $\mathcal{L}[f] = F(s)$ and $\mathcal{L}[g] = G(s)$, and where the convolution of f and g is defined as

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

(a) Given that $\mathcal{L}[f(t)] = F(s)$ and that $b > 0$, find the Laplace Transform of $H(t-b)f(t-b)$ in terms of $F(s)$, where H denotes the unit step function. Hence find the inverse transform of the function,

$$e^{-bs}\frac{s}{s^2+a^2}.$$

[4 marks]

(b) Find the Laplace Transform of the unit impulse at $t = b$, namely, $\delta(t-b)$. [2 marks]

(c) Given the result of part (b), use the convolution theorem to find, by a different approach, the inverse transform of the function given in part (a). [4 marks]

Question 7.

(a) By writing down the definition of the Newton-Raphson iteration scheme for $f(x) = x^2 - 4 = 0$ show that the following formula may be used to find the square roots of 4:

$$x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}.$$

Verify that this formula does find a square root of 4 by evaluating five successive iterates beginning with $x_1 = 3$. [5 marks]

(b) Analyse the approach to the limit $x_\infty = 2$ by setting $x_n = 2 + \epsilon_n$ where ϵ_n is very small, and by evaluating x_{n+1}. [5 marks]

Question 8.

(a) In two separate computations use $N = 4$ and $N = 8$ equally-spaced intervals to approximate the integral,

$$\int_1^3 \sqrt{x}\, dx$$

using the trapezium rule. Solve the integral analytically and find the exact error for each value of N. [4 marks]

The trapezium rule has second order accuracy. Use Richardson's extrapolation formula to obtain a more accurate numerical value of the integral. What is the error for this extrapolate? [1 mark]

(b) A new numerical integration method has been devised to perform numerical integrations, but its order of accuracy is unknown. Therefore the method has been applied for N intervals, $2N$ intervals and $4N$ intervals. Derive the formula for the ratio test to estimate the order of accuracy. [3 marks]

(c) The following data has been obtained, where I_N denotes the numerical approximation to the exact integral, I, using N intervals:

N	I_N
10	1.202503
20	1.122439
40	1.112524

Using the ratio test formula devised in part (b), determine the order of accuracy of the method which produced this data, and then use Richardson's extrapolation on I_{20} and I_{40} to determine an even more accurate estimate of the integral. [2 marks]

Question 9.

The matrix, A, is defined as follows:

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -3 \end{pmatrix}.$$

(a) Find all the eigenvalues and eigenvectors of A. [6 marks]

(b) Hence write down the general solution of the system of ordinary differential equations,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

[2 marks]

(c) What is the solution of the equations given in part (b) if the initial condition is that $(x, y, z) = (2, 0, 2)$ at $t = 0$? [2 marks]

Question 10.

(a) On your standard bicycle training ride of 30 miles, a good day corresponds to when the journey takes less than 90 minutes, otherwise it is considered to be a bad day. Over many such rides you notice that a good day is followed by a good day with probability 0.9, while a bad day is followed by another bad day with the probability 0.3. By creating a tree diagram or otherwise, create a probability transition matrix which describes these probabilities. [3 marks]

(b) Unfortunately, today's ride was quite spectacularly bad. Use matrix multiplication to determine the probabilities associated with the quality of the ride which you might expect after three more rides. [4 marks]

(c) Your intermediate answers to part (b) might possibly suggest what proportion of your rides are likely to be poor in the long-term. By determining the eigenvalues and eigenvectors of the probability transition matrix, determine what your long-term ride characteristics are, i.e. determine the probability of a poor ride on a randomly selected day. [3 marks]