Department of Mechanical Engineering, University of Bath

Mathematics ME10305

Problem Sheet 2 (supplementary) — ODE solutions

This problem sheet contains questions all of which are over and above what is required in the exams. You may therefore treat this sheet as purely optional. Nevertheless, everything that is given here may be completed using what has been taught in Maths 1 and Maths 2, with a few hints and nudges along the way.

Q1. One notation for dy/dt which is sometimes used in textbooks and research papers is Dy. In essence, d/dt and D are directly equivalent to one another and are simply alternative ways of writing down the same thing. Given this, one may try to determine the inverse of D in the following way. Given that

$$\frac{dy}{dt} = f(t) \qquad \Rightarrow \qquad y = c + \int f(t)\,dt,$$

then we may define D^{-1} as follows,

$$Dy = f \qquad \Rightarrow \qquad y = \frac{1}{D}f(t) \qquad = \qquad c + \int f(t)\,dt.$$

In other words, D^{-1} is equivalent to an indefinite integral plus an arbitrary constant.

(a) Now consider the differential equation, $(D+a)y = f(t)$. Rewrite this in the usual way (i.e. $dy/dt+ay = f(t)$) and use the integrating factor approach to find y, not forgetting the arbitrary constant. When this is done, identify which part of your solution forms the Complementary function and which the Particular Integral. What you have written is then the equivalent of

$$y = \frac{1}{D+a}f(t),$$

and it defines the meaning of $(D+a)^{-1}$.

(b) Let us extend the result of Q1a to the following differential equation,

$$\frac{d^2y}{dt^2} + (a+b)\frac{dy}{dt} + aby = f(t).$$

This may also be written as

$$D^2y + (a+b)Dy + aby = f(t), \qquad \text{or} \qquad (D+a)(D+b)y = f(t).$$

If we now set $z = (D+b)y$ then $(D+a)z = f(t)$.

First solve $(D+a)z = f(t)$ for z by applying the result of Q1a directly. Then solve $(D+b)y = z$ to find y. Keep your wits about you on this one — the final answer will involve a double integral.

(c) Now we will modify slightly the answer given in Q1b for the case when $a = b$, which (in the terminology of the lectures) is a repeated-λ case. You should find that some integrals will simplify slightly.

(d) Apply the formula found in Q1b to solve the two equations,

$$y'' + 3y' + 2y = e^t \qquad \text{and} \qquad y'' + 3y' + 2y = e^{-t}.$$

(e) Suppose that we are solving a third order ODE with $f(t)$ on the right hand side. If it can be written in the form,

$$(D+a)(D+b)(D+c)y = f(t),$$

and given the form of the answer Q1b, can you guess what the solution is?

Q2. The aim for this question is to solve $y' + ay = 1$ subject to $y(0) = 0$ using Taylor's series. First, write down a general expression for the Taylor's series about $t = 0$ for the function $y(t)$ — this is *not* the solution because we don't yet know the value of all of the derivatives of y at $t = 0$. However, we may substitute the initial value of y into the governing equation to find $y'(0)$. Now differentiate the governing equation once; this will allow us to find $y''(0)$. Differentate again and hence find $y'''(0)$. The pattern should now be clear. Hence write down the Taylor's series of the solution. Can you identify it?

Q3. This question was devised while I was watching the film, Gravity, en route to India, with only a thin skin of aluminium between me and a quarter of an atmosphere of air at $-50°$C and 500mph six miles above the ground. I am not sure that I like disaster movies while flying! Suppose that Sandra Bullock and George Clooney are stranded in space, 20m apart and stationary relative to each another, i.e. 10m from their centre of gravity (I am assuming that they have the same mass!). How long will it take for gravitational attraction to cause the couple get close enough together that they may grasp each other's hand? So if $x(t)$ is the distance of one of them from their mutual centre of gravity, how long will it take to reduce $x = 10$ to $x = 0.5$ as gravitational attraction draws them together? (Of course, this is being typeset on Valentine's day.) The governing equation is

$$m_1 \frac{d^2x}{dt^2} = -\frac{m_1 m_2 G}{x^2},$$

where $m_1 = m_2 = 60$kg are their masses, and $G = 6.67408 \times 10^{-11}\,\mathrm{N\,m^2kg^{-2}}$ is the gravitational constant. This is a nonlinear second order equation!

(a) Nothing in our lectures hints about how to solve this! However, d^2x/dt^2 is the same as dv/dt where $v = dx/dt$. Use the chain rule to show that

$$\frac{dv}{dt} = v\frac{dv}{dx}.$$

Use this substitution to solve for v in terms of x. Apply the initial condition, namely that at $t = 0$, $x = x_0$ and $v = 0$ (we'll keep the initial separation general for now).

(b) Now that we have v in terms of x, it is possible to solve this by first using the substitution, $x = x_0 \cos^2\theta$, to obtain an equation for θ in terms of t. This equation may be solved to find t in terms of θ. Don't let this worry you, for the whole point is that you need to find the time corresponding to a given distance. Now use $x_0 = 10$ and let $x = 0.5$ (it's probably best to find the corresponding value of θ here); what is the time? So how many days does it take for them to be reunited? (Cue suitable violin music...)

Q4. The Cauchy-Euler equation is a different class of linear ODE, and technically it is known as an equi-dimensional equation. The most general second order version is

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0.$$

There are two ways of solving this equation, the first being to let $y = x^n$ (and then one will eventually be led to an indicial/auxiliary/characteristic equation for n) while the second is to change variables from x to ξ using $x = e^\xi$. Try to solve the equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$$

using each of these two methods. [Note, when attempting the second, we are changing from dy/dx to $dy/d\xi$, and the chain rule will need to be used. Take care with the transformation of the second derivative — the product rule will be needed!]

Suppose now that we wish to solve

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0.$$

The first method given above leads to a repeated value of n and then it isn't obvious how to proceed in this context. So adopt the second method, solve the equation, and this will show how one may proceed when using the otherwise quicker and simpler first method.

D.A.S.R. 14/02/2017