Department of Mechanical Engineering, University of Bath

Mathematics 2 ME10305

Problem Sheet 1 — ODEs

Q1. What is the order of the following equations or systems of equations?

In each case rewrite them in first order form. **Do not** try to solve them!

Are these equations/systems linear or nonlinear, and do they constitute Initial Value Problems or Boundary Value Problems? (Primes denote derivatives with respect to t.)

(a) $y'' + ty = 0$ subject to $y(0) = 0, \quad y'(0) = 1.$

(b) $y''' + y'' - 2yz = 0, \quad z' = ty$ subject to $y(0) = 1, \quad y'(0) = 0, \quad y'(\infty) = 0, \quad z(0) = 0.$

(c) $y'''' + 2(y + y'')^3 y' + y^5 = 1,$ subject to $y = y' = y'' = y''' - 1 = 0$ at $t = 0.$

(d) $x'' + 2x - y = 0, \quad y'' - x + 2y - z = 0, \quad z'' - 3y + 2z = 0,$
subject to $x(0) = 1, \quad x'(0) = 0, \quad y(0) = y'(0) = 0, \quad z(0) = z'(0) = 0.$

(e) $f' = g, \quad g'' + fg' + f'g = 0,$ subject to $f(0) = 0, \quad g(0) = 1, \quad g(\infty) = 0.$

Q2. Solve the following equations by direct integration of both sides.

(a) $y' = \cos t$ subject to $y(0) = 1,$ (b) $y' = e^{2t} + 1$ subject to $y(1) = 1.$

Q3. Use separation of variables to find the solutions to the following ODEs,

(a) $\dfrac{dy}{dt} = \dfrac{4t}{y}, \quad y(0) = 1,$ (b) $\dfrac{dy}{dt} = 3t^2 y, \quad y(0) = 1,$

(c) $\dfrac{dy}{dt} = t(1 + y^2) \quad y(0) = 1,$ (d) $\dfrac{dy}{dt} = t^2(1 - y^2), \quad y(0) = 2,$

(e) $\dfrac{dy}{dt} = y - y^2, \quad y(0) = 2,$ (f) $t^2 \dfrac{dy}{dt} = y - t^3 y, \quad y(1) = 1,$

(g) $\dfrac{d^2y}{dt^2} = \dfrac{1}{t}\dfrac{dy}{dt}, \quad y(0) = 1, \; y'(1) = 2.$

Q4. Find the Integrating Factor and hence solve the following 1st order equations.

(a) $\dfrac{dy}{dt} + \dfrac{y}{t} = 1,$ (b) $\dfrac{dy}{dt} - \dfrac{y}{t} = 1,$ (c) $\dfrac{dy}{dt} + \dfrac{3y}{t} = t^{-2},$ (d) $\dfrac{dy}{dt} + 2ty = 2t,$

(e) $\dfrac{dy}{dt} + y \cot t = 1,$ (f) $\dfrac{dy}{dt} + \dfrac{1 + 2t}{t} y = \dfrac{1}{t},$ (g) $\dfrac{dy}{dt} + 4t^3 y = t^3$ (h) $t\dfrac{dy}{dt} + (t + 1)y = t^2.$

Continued...

Q5. The following differential equation

$$\frac{dy}{dt} = y^3 - y$$

falls into two different categories. First, it is of variables-separable type, and second it is an example of what is known as a Bernoulli equation.

(i) Use separation of variables, followed by partial fractions to find the solution subject to the initial condition that $y = 1/\sqrt{2}$ when $t = 0$.

(ii) Solve the equation for y again by first using the substitution, $y = z^{-1/2}$ where $z = z(t)$ is a new dependent variable — you will need to use the chain rule for this to find a formula for dz/dt in terms of dy/dt. This substitution should then give you a linear equation for z which may be solved.

Q6. The general form for Bernoulli's equation is

$$\frac{dy}{dt} + P(t)y^n + Q(t)y = 0,$$

where $P(t)$ and $Q(t)$ are given functions.

(i) Use the substitution $y = z^\alpha$, where z is a function of time and where α is constant to be found. After substitution, determine that value of α which reduces the equation for z into one of first order linear form.

(ii) You are now required to solve the ODE,

$$\frac{dy}{dt} + y - y^{-1} = 0, \qquad \text{subject to} \qquad y = 2 \text{ at } t = 0.$$

Use your general Bernoulli result to reduce the ODE to first order linear form and solve it. Check your answer for y by substituting it back into the above equation.

(iii) The above equation may also be solved using separation of variables. Please do it this way as well.

Q7. Another category of ODE could be called equidimensional. This is an example:

$$\frac{dy}{dx} = \frac{2y^2 + x^2}{2xy}.$$

The method of solution is to substitute $y(x) = x\,v(x)$ to form an ODE for $v(x)$. The resulting equation should then be solvable using separation of variables. Solve the above equation subject to the initial condition, $y = 1$ when $x = 1$. Then check that your solution satisfies the original ODE. [You may also attempt Q4a using the same idea.]

Please note that all of the above could form at least part of an exam question. I will not ask for the derivation required in Q6i. For equations of Bernoulli type and for those in equidimensional form I will give the required substitutions (as I have in Q5ii and Q7).

D.A.S.R. 8/1/2020