Department of Mechanical Engineering, University of Bath

Mathematics ME10305

Problem Sheet 2 (supplementary) — ODE solutions

This problem sheet contains questions all of which are over and above what is required in the exams. You may therefore treat this sheet as purely optional. Nevertheless, everything that is given here may be completed using what has been taught in Maths 1 and Maths 2, with a few hints and nudges along the way.

Q1. One notation for dy/dt which is sometimes used in textbooks and research papers is Dy. In essence, d/dt and D are directly equivalent to one another and are simply alternative ways of writing down the same thing. Given this, one may try to determine the inverse of D in the following way. Given that

$$\frac{dy}{dt} = f(t) \qquad \Rightarrow \qquad y = c + \int f(t)\,dt,$$

then we may define D^{-1} as follows,

$$Dy = f \qquad \Rightarrow \qquad y = \frac{1}{D}f(t) \qquad = \qquad c + \int f(t)\,dt.$$

In other words, D^{-1} is equivalent to an indefinite integral plus an arbitrary constant.

(a) Now consider the differential equation, $(D+a)y = f(t)$. Rewrite this in the usual way (i.e. $dy/dt + ay = f(t)$) and use the integrating factor approach to find y, not forgetting the arbitrary constant. When this is done, identify which part of your solution forms the Complementary function and which the Particular Integral. What you have written is then the equivalent of

$$y = \frac{1}{D+a}f(t),$$

and it defines the meaning of $(D+a)^{-1}$.

ANS. The integrating factor is $\exp[\int a\,dt]$ which is e^{at}. Therefore

$$\frac{dy}{dt} + ay = f(t) \qquad \text{becomes} \qquad e^{at}\Big(\frac{dy}{dt} + ay\Big) = e^{at}f(t) \qquad \Rightarrow \qquad \frac{d}{dt}\Big(e^{at}y\Big) = e^{at}f(t).$$

The left hand side of this latest equation is an exact differential, and therefore we may integrate to obtain,

$$e^{at}y = c + \int e^{at}f(t)\,dt \qquad \Rightarrow \qquad y = c\,e^{-at} + e^{-at}\int e^{at}f(t)\,dt.$$

Clearly, the first part of the solution is the Complementary Function, while the one involving the integral is the Particular Integral. Therefore,

$$\frac{1}{D+a}f(t) = c\,e^{-at} + e^{-at}\int e^{at}f(t)\,dt.$$

(b) Let us extend the result of Q1a to the following differential equation,

$$\frac{d^2y}{dt^2} + (a+b)\frac{dy}{dt} + aby = f(t).$$

This may also be written as

$$D^2y + (a+b)Dy + aby = f(t), \qquad \text{or} \qquad (D+a)(D+b)y = f(t).$$

If we now set $z = (D+b)y$ then $(D+a)z = f(t)$.

First solve $(D+a)z = f(t)$ for z by applying the result of Q1a directly. Then solve $(D+b)y = z$ to find y. Keep your wits about you on this one — the final answer will involve a double integral.

ANS. First we solve $(D+a)z = f$ using the result of Q1a. We have,

$$z = c_1\, e^{-at} + e^{-at} \int e^{at} f(t)\, dt. \tag{1}$$

Now we solve for $(D+b)y = z$. We have,

$$y = c_2\, e^{-bt} + e^{-bt} \int e^{bt} z(t)\, dt. \tag{2}$$

Now we shall substitute the expression for z given in (1) into the solution for y in equation (2):

$$\begin{aligned}
y &= c_2\, e^{-bt} + e^{-bt} \int e^{bt} \Big[c_1\, e^{-at} + e^{-at} \int e^{at} f(t)\, dt \Big]\, dt \\
&= c_2\, e^{-bt} + c_1 e^{-bt} \int e^{(b-a)t}\, dt + e^{-bt} \int e^{(b-a)t} \Big[\int e^{at} f(t)\, dt \Big]\, dt \\
&= c_2\, e^{-bt} + c_1 e^{-bt} \Big[\frac{e^{(b-a)t}}{b-a} \Big] + e^{-bt} \int e^{(b-a)t} \Big[\int e^{at} f(t)\, dt \Big]\, dt \\
&= c_2\, e^{-bt} + c_1^* e^{-at} + e^{-bt} \int e^{(b-a)t} \Big[\int e^{at} f(t)\, dt \Big]\, dt,
\end{aligned} \tag{3}$$

where c_1^* is a redefined arbitrary constant, Here, the terms involving c_1 and c_2, the arbitrary constants, form the Complementary Function, and the term with the integrals is the Particular Integral. This expression is correct for any choice of the constants, a and b, but only if they are different.

(c) Now we will modify slightly the answer given in Q1b for the case when $a = b$, which (in the terminology of the lectures) is a repeated-λ case. You should find that some integrals will simplify slightly.

ANS. This is not simply a case of replacing b by a in Equation (3), above. Rather, we need to rework the analysis which leads to (3), as follows.

$$\begin{aligned}
y &= c_2\, e^{-at} + e^{-at} \int e^{at} \Big[c_1\, e^{-at} + e^{-at} \int e^{at} f(t)\, dt \Big]\, dt \\
&= c_2\, e^{-bt} + c_1 \int 1\, dt + e^{-at} \int \Big[\int e^{at} f(t)\, dt \Big]\, dt \\
&= c_2\, e^{-at} + c_1 t e^{-at} + e^{-at} \int \Big[\int e^{at} f(t)\, dt \Big]\, dt.
\end{aligned} \tag{4}$$

(d) Apply the formula found in Q1b to solve the two equations,

$$\text{(a)} \quad y'' + 3y' + 2y = e^{t} \qquad \text{and} \qquad \text{(b)} \quad y'' + 3y' + 2y = e^{-t}.$$

ANS. For both equations we may use $a = 1$ and $b = 2$. For both equations we obtain Complementary functions of the form, Ae^{-2t} and Be^{-t}, where I have reverted to the notation used in the lectures for the arbitrary constants.

For the Particular Integral for the first equation we have,

$$\begin{aligned}
y_{\rm pi} &= e^{-2t}\int e^t\Big[\int e^t \times e^t\,dt\Big]\,dt && \text{subst. into (3)}\\
&= e^{-2t}\int e^t\Big[\int e^{2t}\,dt\Big]\,dt && \text{on mutiplying}\\
&= e^{-2t}\int e^t\Big[\tfrac{1}{2}e^{2t}\Big]\,dt && \text{on integrating}\\
&= \tfrac{1}{2}e^{-2t}\int e^{3t}\,dt && \text{on multiplying}\\
&= \tfrac{1}{2}e^{-2t}\times\tfrac{1}{3}e^{3t} && \text{on integrating}\\
&= \tfrac{1}{6}e^t.
\end{aligned}$$

Hence the case (a) solution is,

$$y = Ae^{-2t} + Be^{-t} + \tfrac{1}{6}e^t.$$

For case (b) we have,

$$\begin{aligned}
y_{\rm pi} &= e^{-2t}\int e^t\Big[\int e^t \times e^{-t}\,dt\Big]\,dt && \text{subst. into (3)}\\
&= e^{-2t}\int e^t\Big[\int 1\,dt\Big]\,dt\\
&= e^{-2t}\int e^t\,t\,dt\\
&= e^{-2t}\Big[(t-1)e^t\Big]\\
&= (t-1)e^{-t}.
\end{aligned}$$

If we had solved this equation using the standard CF/PI approach, then the PI would have been just te^{-t}, and this would also have been true if we had used $a = 2$ and $b = 1$ above. The extra $-e^{-t}$ may be swallowed up in the CF, given that part of the CF is Be^{-t} where B is arbitrary.

(e) Suppose that we are solving a third order ODE with $f(t)$ on the right hand side. If it can be written in the form,

$$(D+a)(D+b)(D+c)y = f(t),$$

and given the form of the answer Q1b, can you guess what the solution is?

ANS. The solution is,

$$y = c_1e^{-at} + c_2e^{-bt} + c_3e^{-ct} + e^{-ct}\int e^{(c-b)t}\Big[\int e^{(b-a)t}\Big[\int e^{at}f(t)\,dt\Big]\,dt\Big]\,dt.$$

Q2. The aim for this question is to solve $y' + ay = 1$ subject to $y(0) = 0$ using Taylor's series. First, write down a general expression for the Taylor's series about $t = 0$ for the function $y(t)$ — this is *not* the solution because we don't yet know the value of all of the derivatives of y at $t = 0$. However, we may substitute the initial value of y into the governing equation to find $y'(0)$. Now differentiate the governing equation once; this will allow us to find $y''(0)$. Differentate again and hence find $y'''(0)$. The pattern should now be clear. Hence write down the Taylor's series of the solution. Can you identify it?

ANS. The required Taylor's series is

$$y(t) = y(0) + \frac{y'(0)}{1}t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \cdots.$$

Successive derivatives of the governing equation, $y' + ay = 1$, are,

$$y'' + ay' = 0, \qquad y''' + ay'' = 0, \qquad y^{(4)} + ay''' = 0, \qquad y^{(5)} + ay^{(4)} = 0, \cdots.$$

If $y(0) = 0$, then the ODE gives us that $y'(0) = 1$.

The derivative of the ODE now tells us that $y''(0) = -ay'(0) = -a$.

The next derivative yields $y'''(0) = +a^2$ and so on with $-a^3$ and $+a^4$ for the next two derivatives at $t = 0$. Hence the solution may be written as,

$$y = t - \frac{at^2}{2!} + \frac{a^2t^3}{3!} - \frac{a^3t^4}{4!} + \frac{a^4t^5}{5!} + \cdots.$$

We know that

$$e^{-at} = 1 - at + \frac{a^2t^2}{2!} - \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!} + \cdots,$$

and therefore our Taylor's series solution represents,

$$y = \frac{1 - e^{-at}}{a}.$$

Q3. This question was devised while I was watching the film, Gravity, en route to India, with only a thin skin of aluminium between me and a quarter of an atmosphere of air at $-50°$C and 500mph six miles above the ground. I am not sure that I like disaster movies while flying! Suppose that Sandra Bullock and George Clooney are stranded in space, 20m apart and stationary relative to each another, i.e. 10m from their centre of gravity (I am assuming that they have the same mass!). How long will it take for gravitational attraction to cause the couple get close enough together that they may grasp each other's hand? So if $x(t)$ is the distance of one of them from their mutual centre of gravity, how long will it take to reduce $x = 10$ to $x = 0.5$ as gravitational attraction draws them together? (Of course, this is being typeset on Valentine's day.) The governing equation is

$$m_1 \frac{d^2x}{dt^2} = -\frac{m_1 m_2 G}{x^2},$$

where $m_1 = m_2 = 60$kg are their masses, and $G = 6.67408 \times 10^{-11}\,\mathrm{N\,m^2kg^{-2}}$ is the gravitational constant. This is a nonlinear second order equation!

(a) Nothing in our lectures hints about how to solve this! However, d^2x/dt^2 is the same as dv/dt where $v = dx/dt$. Use the chain rule to show that

$$\frac{dv}{dt} = v\frac{dv}{dx}.$$

Use this substitution to solve for v in terms of x. Apply the initial condition, namely that at $t = 0$, $x = x_0$ and $v = 0$ (we'll keep the initial separation general for now) For convenience, use $\lambda^2 = Gm_2$ in your workings.

ANS. First of all, I have not said anything about what x represents. The given equation, should one look it up on Wikipedia for example, will state that x is the distance of one mass from the other, and therefore we need to be a little careful at the outset to define x correctly. Perhaps the easiest way to do this is to let George and Sandra be at the two locations $x = y$ and $x = -y$ so that $x = 0$ represents their combined centre of gravity and, given the 'fact' that they have the same mass, it is also the point to which they will converge eventually. This means that their separation will be $2x$, and it is this distance which we need to use in the law of graviational attraction. From the point of view of the person whose mass is m_1, the equation of motion is

$$m_1 \frac{d^2x}{dt^2} = -\frac{m_1 m_2 G}{(2x)^2},$$

where x is the distance from the origin.

We know that

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\Big(\frac{dx}{dt}\Big) = \frac{dv}{dt},$$

and we wish to change the t-derivative to an x-derivative. The chain rule gives us,

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}.$$

Hence the governing equation becomes,

$$v\frac{d^2v}{dx^2} = -\frac{m_2 G}{4x^2},$$

which is of variables-separable form, and where I have cancelled the m_1 coefficients on both sides. So we need to solve,

$$v\frac{d^2v}{dx^2} = -\frac{\lambda^2}{4x^2}.$$

On separating the variables we have,

$$v\,dv = -\frac{\lambda^2}{4x^2}\,dx,$$

which, upon integration yields,

$$\tfrac{1}{2}v^2 = \frac{\lambda^2}{4x} + c.$$

When $t = 0$ we have $x = x_0 = 10$ and $v = 0$, and therefore $c = -\lambda^2/4x_0$. Therefore the solution so far is,

$$v^2 = \frac{\lambda^2}{2}\Big(\frac{1}{x} - \frac{1}{x_0}\Big),$$

and hence

$$v = -\frac{\lambda}{\sqrt{2}}\Big(\frac{1}{x} - \frac{1}{x_0}\Big)^{1/2}.$$

When taking the square roots, the negative sign has been taken because x will decrease in time and hence $v < 0$.

(b) Now that we have v in terms of x, it is possible to solve this by first using the substitution, $x = x_0\cos^2\theta$, to obtain an equation for θ in terms of t. This equation may be solved to find t in terms of θ. Don't let this worry you, for the whole point is that you need to find the time corresponding to a given distance. Now use $x_0 = 10$

and let $x = 0.5$ (it's probably best to find the corresponding value of θ here); what is the time? So how many days does it take for them to be reunited? (Cue suitable violin music...)

ANS. Now we revert to x, and therefore we have,

$$\frac{dx}{dt} = -\frac{\lambda}{\sqrt{2}}\Big(\frac{1}{x} - \frac{1}{x_0}\Big)^{1/2}. \tag{1}$$

Following the hint, we will change variable from x to θ using $x = x_0 \cos^2\theta$. So

$$\frac{dx}{dt} = -2x_0 \sin\theta\cos\theta\,\frac{d\theta}{dt}.$$

Hence the equation becomes,

$$\begin{aligned}-2x_0\sin\theta\cos\theta\,\frac{d\theta}{dt} &= -\frac{\lambda}{\sqrt{2x_0}}\Big[\frac{1}{\cos^2\theta} - 1\Big]^{1/2}\\ &= -\frac{\lambda}{\sqrt{2x_0}}\Big[\frac{1-\cos^2\theta}{\cos^2\theta}\Big]^{1/2}\\ &= -\frac{\lambda}{\sqrt{2x_0}}\frac{\sin\theta}{\cos\theta}.\end{aligned}$$

Therefore the equation may be tidied up to give,

$$\cos^2\theta\frac{d\theta}{dt} = \frac{\lambda}{2\sqrt{2}\,x_0^{3/2}},$$

or, even better,

$$\tfrac{1}{2}(1+\cos 2\theta)\frac{d\theta}{dt} = \frac{\lambda}{2\sqrt{2}\,x_0^{3/2}}.$$

This is another separation-of-variables equation, and we may integrate to get,

$$\tfrac{1}{2}\Big(\theta + \tfrac{1}{2}\sin 2\theta\Big) = \frac{\lambda}{2\sqrt{2}\,x_0^{3/2}}t + c.$$

The initial condition is that, at $t = 0$ we have $x = x_0$ and hence $\cos\theta = 1$ or $\theta = 0$. Therefore $c = 0$. Therefore we may now write the final solution (in terms of θ) as

$$t = 2\Big(\frac{x_0^3}{2Gm_2}\Big)^{1/2}\Big(\theta + \tfrac{1}{2}\sin 2\theta\Big).$$

Although we cannot rearrange this equation to get θ (and hence x) in terms of t, it doesn't matter here, for we need to find t when $x = 0.5$. This translates into when $\cos^2\theta = 0.5/10$, i.e. $\theta = 1.34528$ radians. Hence

$$t = 2\times 1.563228\times\Big(\frac{1000}{2\times 60\times 6.67408\times 10^{-11}}\Big)^{1/2} = 1\,194\,755s.$$

This is just under 12 days and 19 hours. At that point in time their relative speed would be about 0.123mm/s (using Eq. (1)), or just over the width of the human hair per second, so it would be a very gentle meeting of fingertips. Nice.

However, given that the human body can survive at most only three days without food and water, this tale has a very sad ending....no violin music....not that it can be heard in space....

If this had been a purely theoretical problem involving point masses instead of film stars, then the point masses would have collided only about 89 minutes later, when $\theta = \frac{1}{2}\pi$ and at an infinite velocity!

Q4. The Cauchy-Euler equation is a different class of linear ODE, and technically it is known as an equi-dimensional equation. The most general second order version is

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0.$$

There are two ways of solving this equation, the first being to let $y = x^n$ (and then one will eventually be led to an indicial/auxiliary/characteristic equation for n) while the second is to change variables from x to ξ using $x = e^\xi$. Try to solve the equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$$

using each of these two methods. [Note, when attempting the second, we are changing from dy/dx to $dy/d\xi$, and the chain rule will need to be used. Take care with the transformation of the second derivative — the product rule will be needed!]

Suppose now that we wish to solve

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0.$$

The first method given above leads to a repeated value of n and then it isn't obvious how to proceed in this context. So adopt the second method, solve the equation, and this will show how one may proceed when using the otherwise quicker and simpler first method.

Ans. For $x^2y'' + 4xy' + 2y = 0$, we let $y = x^n$. This yields,

$$x^2\Big[n(n-1)x^{n-2}\Big] + 4x\Big[nx^{n-1}\Big] + 2x^n = 0$$

$$\Big[(n^2 - n) + 4n + 2\Big]x^n = 0.$$

This shows why the given substitution works: the only function which, when it is differentiated m times and then is multiplied by x^m, gives the same function is a power of x. The indicial equation for n is now

$$n^2 + 3n + 2 = 0,$$

and we find that $n = -1, -2$. Hence the solution follows in the same way as we have for constant-coefficient equations:

$$y = Ax^{-1} + Bx^{-2}. \tag{4}$$

NOTE: we will meet these in Modelling Techhniques 2 in semester 2 of year 2.

Alternatively, if we substitute $x = e^\xi$, then we need to change from x-derivatives to ξ-derivatives. Therefore,

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = e^{-\xi}\frac{dy}{d\xi}.$$

This is more easily dealt with if we multiply both sides (i.e. leftmost and rightmost) by e^ξ (i.e. x). We get,

$$x\frac{dy}{dx} = \frac{dy}{d\xi},$$

which is very useful. For the second derivatives, note first that,

$$\frac{d^2y}{d\xi^2} = \frac{d}{d\xi}\Big(\frac{dy}{d\xi}\Big) = x\frac{d}{dx}\Big(x\frac{dy}{dx}\Big) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}.$$

Hence,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}.$$

Therefore the given equation transforms as follows,

$$\left(\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\right) + 4\frac{dy}{d\xi} + 2y = 0,$$

and hence

$$\frac{d^2y}{d\xi^2} + 3\frac{dy}{d\xi} + 2y = 0.$$

Therefore our substitution transforms a Cauchy-Euler equation into a constant-coefficient equation. Also nice, and also extremely useful as we will see. The solution of this latest equation is,

$$y = Ae^{-\xi} + Be^{-2\xi},$$

which is identical to equation (4), above.

When we consider the final equation in the question, $x^2y'' + 5xy' + 4y = 0$, the $y = x^n$ trial solutions yields the repeated pair, $n = -2, -2$. Therefore let us see what the coordinate transformation gives us. Using the above results we obtain,

$$\frac{d^2y}{d\xi^2} + 4\frac{dy}{d\xi} + 4y = 0.$$

Letting $y = e^{\lambda\xi}$ also yields $\lambda = -2, -2$, and therefore the solution is

$$y = (A + B\xi)e^{-2\xi}.$$

When we revert to x, this becomes,

$$y = (A + B\ln x)x^{-2}.$$

This gives us the clue for how one progresses quickly when repeated roots of the indicial equation are encountered when solving a Cauchy-Euler equation.