Department of Mechanical Engineering, University of Bath

Mathematics ME10305

Problem Sheet 2 — ODE solutions — linear, constant coefficient, ODEs

Q1. First find the general solution of the following homogeneous equations. Then find the solution which satisfies $y(0) = 1$ and $y'(0) = 0$ (additionally $y''(0) = 0$ for third and fourth order equations and $y'''(0) = 0$ for fourth order equations).

(a) $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 4y = 0$; (b) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 0$; (c) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = 0$; (d) $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 29y = 0$;

(e) $\dfrac{d^3y}{dt^3} + 2\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 2y = 0$; (f) $\dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} - 2y = 0$; (g) $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = 0$;

(h) $\dfrac{d^4y}{dt^4} + 4y = 0$; (i) $\dfrac{d^4y}{dt^4} + 5\dfrac{d^2y}{dt^2} + 4y = 0$; (j) $\dfrac{d^4y}{dt^4} + 2\dfrac{d^2y}{dt^2} + y = 0$.

A1. In all cases we set $y = e^{\lambda t}$ to obtain the Auxiliary (or Indicial or Characteristic) equation for λ. When this equation is solved, the standard case is when all the possible λ values are different. Increased difficulties, and an increased length of analysis, arise when there are repeated values of λ.

In what follows we'll consider the general solutions first, and afterwards the boundary conditions are applied to each general solution in turn.

(a) On setting $y = e^{\lambda t}$ into the ODE yields,

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 4e^{\lambda t} = 0 \qquad \Rightarrow \qquad \Big[\lambda^2 + 5\lambda + 4\Big]e^{\lambda t} = 0.$$

We may remove the $e^{\lambda t}$ as it is never zero, and therefore the polynomial in λ must be zero. We have

$$\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) = 0.$$

Hence λ is either -1 or -4. In these contexts we take both, and set

$$y = Ae^{-t} + Be^{-4t}$$

where both A and B are arbitrary. Values of A and B may only be found if two boundary conditions are given, but I haven't done that in this question.

(b) Following the same procedure yields

$$0 = \lambda^2 + 4\lambda + 4 = \Big(\lambda + 2\Big)^2.$$

Therefore we have a repeated root: $\lambda = -2, -2$. Therefore the solution is

$$y = (At + B)e^{-2t}.$$

(c) Following the same procedure we get

$$0 = \lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4.$$

Therefore $(\lambda+1)^2 = -4$ and hence $\lambda = -1 \pm 2j$. These values for λ could also be obtained using the standard formula for the solution of a quadratic, but here I have simply completed the square, and taken it from there....

On using these values, the solution is,

$$y = Ae^{(-1+2j)t} + Be^{(-1-2j)t}.$$

Given that $e^{2jt} = \cos 2t + j \sin 2t$, and that $e^{-2jt} = \cos 2t - j \sin 2t$, the solution may be written in the form,

$$y = e^{-t}\Big[A^* \cos 2t + B^* \sin 2t\Big],$$

where A^* and B^* are new arbitrary constants.

(d) Following the same procedure we get

$$0 = \lambda^2 - 4\lambda + 29 = \Big(\lambda - 2\Big)^2 + 25.$$

Hence $\lambda = 2 \pm 5j$. As in part (c), we may write the solution in the form,

$$y = e^{2t}\Big[A \cos 5t + B \sin 5t\Big].$$

(e) In this case the Auxiliary Equation is

$$\lambda^3 + 2\lambda^2 + \lambda + 2 = 0.$$

This may be factorised,

$$(\lambda+2)(\lambda^2+1) = 0,$$

and therefore $\lambda = -2,\ \pm j$. The solution of the equation is

$$y = Ae^{-2t} + B\cos t + C \sin t.$$

(f) This one yields,

$$0 = \lambda^3 + \lambda^2 - 2 = (\lambda-1)(\lambda^2 + 2\lambda + 2) = (\lambda-1)\Big((\lambda+1)^2 + 1\Big).$$

Therefore $\lambda = 1,\ -1 \pm j$, and the solution is

$$y = Ae^t + e^{-t}\Big(B\cos t + C\sin t\Big).$$

(g) This time we get,

$$0 = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3.$$

Recognise the Binomial coefficients? This is a three-times repeated root, $\lambda = 1, 1, 1$, and the solution is

$$y = (At^2 + Bt + C)e^t.$$

(h) For this one we get,

$$\lambda^4 + 4 = 0 \qquad \Rightarrow \qquad \lambda^4 = -4 \qquad \Rightarrow \qquad \lambda^2 = \pm 2j \qquad \Rightarrow \qquad \lambda = \pm 1 \pm j,$$

where all four possible choices of sign may be taken. The solution may be written as

$$y = e^{t}(A\cos t + B\sin t) + e^{-t}(C\cos t + D\sin t).$$

Given that the exponentials multiply the same types of sinusoid, it is also possible to write the solution in the form,

$$y = A^{*}\cosh t\cos t + B^{*}\sinh t\sin t + C^{*}\cosh t\sin t + D^{*}\sinh t\cos t.$$

(i) We get,

$$\lambda^4 + 5\lambda^2 + 4 = 0,$$

which may be factorised to yield,

$$(\lambda^2 + 1)(\lambda^2 + 4) = 0.$$

Therefore $\lambda^2 = -1, -4$ and so,

$$\lambda = \pm j,\ \pm 2j.$$

The general solution is

$$y = A\cos t + B\sin t + C\cos 2t + D\sin 2t.$$

(j) We get,

$$\lambda^4 + 2\lambda^2 + 1 = 0,$$

which may be factorised to yield,

$$(\lambda^2 + 1)^2 = 0.$$

Therefore $\lambda^2 = -1, -1$ and so $\lambda = \pm j, \pm j$. The general solution is, therefore,

$$y = (A + Bt)\cos t + (C + Dt)\sin t.$$

Now to find the solutions corresponding to the given initial conditions.

(a) The general solution is $y = Ae^{-t} + Be^{-4t}$ and this needs to satisfy $y(0) = 1$ and $y'(0) = 0$. First we find that, $y' = -Ae^{-t} - 4Be^{-4t}$. Hence

$$A + B = 1, \qquad \text{and} \qquad -A - 4B = 0.$$

Therefore $A = \frac{4}{3}$ and $B = -\frac{1}{3}$, and the final solution is

$$y = \tfrac{4}{3}e^{-t} - \tfrac{1}{3}e^{-4t}.$$

(b) The general solution is $y = (At + B)e^{-2t}$. Hence

$$y' = (A - 2B - 2At)e^{-2t}.$$

Application of the initial conditions gives,

$$B = 1 \qquad \text{and} \qquad A - 2B = 0.$$

Hence $A = 2$ and $B = 1$ and the final solution is,

$$y = (1 + 2t)e^{-2t}.$$

(c) The general solution is $y = e^{-t}\Big[A\cos 2t + B\sin 2t\Big]$, where I have removed the asterisks. Application of the initial conditions yields,

$$A = 1 \qquad \text{and} \qquad 2B - A = 0.$$

Hence $A = 1$ and $B = \frac{1}{2}$. Therefore the solution is

$$y = e^{-t}\Big[\cos 2t + \tfrac{1}{2}\sin 2t\Big].$$

(d) The general solution is $y = e^{2t}\Big[A\cos 5t + B\sin 5t\Big]$. We will eventually find that

$$y = e^{2t}\Big[\cos 5t - \tfrac{2}{5}\sin 5t\Big].$$

(e) The general solution is $y = Ae^{-2t} + B\cos t + C\sin t$. We now have to satisfy $y(0) = 1$, $y'(0) = 0$ and $y''(0) = 0$. Successive differentiation is straightforward, and the three conditions give us,

$$A + B = 1, \qquad -2A + C = 0, \qquad 4A - B = 0.$$

The solutions of these three equations are

$$A = \tfrac{1}{5}, \qquad B = \tfrac{4}{5}, \qquad C = \tfrac{2}{5}.$$

Hence the final solution is

$$y = \tfrac{1}{5}e^{-2t} + \tfrac{4}{5}\cos t + \tfrac{2}{5}\sin t.$$

Note that, if one were to expand this solution in a Taylor's series, then we would get,

$$y \simeq 1 - \tfrac{1}{3}t^3 + \text{terms in } t^4 \text{ etc.},$$

which is another way of showing that the initial conditions have been satisfied.

(f) The general solution is $y = Ae^t + e^{-t}\Big(B\cos t + C\sin t\Big)$. Finding successive derivative is now becoming more time-consuming. We find firstly that,

$$A + B = 1, \qquad A + C - B = 0, \qquad A - 2C = 0.$$

Therefore $A = \frac{2}{5}$, $B = \frac{3}{5}$ and $C = \frac{1}{5}$, and hence the final solution is,

$$y = \tfrac{2}{5}e^t + e^{-t}\Big(\tfrac{3}{5}\cos t + \tfrac{1}{5}\sin t\Big).$$

(g) The general solution is $y = (At^2 + Bt + C)e^t$. The solution which satisfies the initial conditions is,

$$y = (1 - t + \tfrac{1}{2}t^2)e^t.$$

(h) The general solution in this case is $y = e^t(A\cos t + B\sin t) + e^{-t}(C\cos t + D\sin t)$. Now we have to satisfy the four initial conditions, $y(0) = 1$ and $y'(0) = y''(0) = y'''(0) = 0$. There is an awful lot of algebra now, and clearly such a question is too long for an exam question, but this is a problem sheet, so we'll press on! The four equations for the constants are,

$$A + C = 1, \qquad A + B - C + D = 0, \qquad B - D = 0, \qquad B - A + C + D = 0.$$

These may be solved to obtain,

$$A = C = \tfrac{1}{2} \qquad \text{and} \qquad B = D = 0.$$

Therefore the solution is

$$\begin{aligned} y &= \tfrac{1}{2}(e^t + e^{-t})\cos t \\ &= \cosh t \cos t. \end{aligned}$$

(i) The general solution is $y = A\cos t + B\sin t + C\cos 2t + D\sin 2t$. The algebraic equations for the constants are

$$A + C = 1, \qquad B + 2D = 0, \qquad -A - 4C = 0, \qquad -B - 8D = 0.$$

These give,

$$A = \tfrac{4}{3}, \qquad B = 0, \qquad C = -\tfrac{1}{3}, \qquad D = 0.$$

Hence the final solution is

$$y = \tfrac{4}{3}\cos t - \tfrac{1}{3}\cos 2t.$$

Again, check the Taylor's series for this solution and you'll find that the first power of t after the leading term, 1, is a t^4 term.

(j) The general solution is $y = (A + Bt)\cos t + (C + Dt)\sin t$. We find that,

$$A = 1, \qquad B + C = 0, \qquad 2D - A = 0, \qquad 3B + C = 0.$$

Therefore,

$$A = 1, \qquad B = C = 0, \qquad D = \tfrac{1}{2}.$$

The final solutions is,

$$y = \cos t + \tfrac{1}{2}t\sin t.$$

Q2. Find the general solution of the following inhomogeneous equations.

(a) $\dfrac{d^2y}{dt^2} + 9y = f(t)$ where $f(t)$ takes the following forms: (i) e^{at}, (ii) t^3, (iii) $\cos at$, (iv) $\cos 3t$.

(b) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = f(t)$ where $f(t)$ takes the following forms: (i) e^{at}, (ii) t^2, (iii) $\cos at$.

(c) $\dfrac{d^2y}{dt^2} - 7\dfrac{dy}{dt} + 12y = f(t)$ where $f(t)$ takes the following forms: (i) e^{2t}, (ii) e^{3t}, (iii) t^2 (iv) $\cos at$.

(d) $\dfrac{d^3y}{dt^3} + 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + y = t^3e^{-t}$. (Do this first using the standard way, and then by using the substitution, $y(t) = z(t)e^{-t}$.)

A2. For inhomogeneous equations it is essential to find the Complementary Function first, since otherwise one might waste a lot of time. Comparison of the functions forming the Complementary Function with the inhomogeneous forcing terms will guide our choice of substitutions for the determination of the Particular Integral. **This is very very important to note.**

(a) The Complementary function in this case is $y_{\text{cf}} = A\cos 3t + B\sin 3t$. Given the possible right hand sides quoted in the question, the only nasty one is the $\cos 3t$, as this is contained within the CF. The others are standard case substitutions.

(i) We simply let $y_{\text{pi}} = Ce^{at}$. Substitution yields,

$$(a^2 + 9)Ce^{at} = e^{at} \qquad \Rightarrow \qquad C = \frac{1}{a^2 + 9}.$$

Therefore the Particular Integral is

$$y_{\text{pi}} = \frac{e^{at}}{a^2+9}.$$

The general solution is

$$y = y_{\text{cf}} + y_{\text{pi}} = A\cos 3t + B\sin 3t + \frac{e^{at}}{a^2+9}.$$

(ii) For the polynomial we set $y_{\text{pi}} = Ct^3 + Dt^2 + Et + F$, although it is possible to work out that D and F are zero by observing that the ODE only has a second derivative. Proceeding with the full expression for y_{pi}, we get

$$[6Ct + 2D] + 9[Ct^3 + Dt^2 + Et + F] = t^3.$$

Equating like coefficients gives us,

$$9C = 1, \qquad 9D = 0, \qquad 6C + 9E = 0, \qquad 2D + 9F = 0.$$

Hence

$$C = \tfrac{1}{9}, \qquad D = 0, \qquad E = -\tfrac{2}{27}, \qquad F = 0.$$

The PI is

$$y_{\text{pi}} = \tfrac{1}{9}t^3 - \tfrac{2}{27}t.$$

The full solution is

$$y = y_{\text{cf}} + y_{\text{pi}} = A\cos 3t + B\sin 3t + \tfrac{1}{9}t^3 - \tfrac{2}{27}t.$$

(iii) For a cosine forcing term we would normally need to use both a cosine and a sine for the PI, but there is no single derivative in this case and just a cosine will work fine. Therefore we set $y_{\text{pi}} = C\cos at$. Omitting details of the analysis, which isn't much at all, we get

$$y_{\text{pi}} = \frac{\cos at}{9-a^2}.$$

Clearly we expect some trouble when $a = 3$, as the denominator will be zero — this is part (iv) of the present question. The final solution is

$$y = y_{\text{cf}} + y_{\text{pi}} = A\cos 3t + B\sin 3t + \frac{\cos at}{9-a^2}.$$

(iv) This is a special case where the forcing term is identical to one of the terms in the Complementary Function. We may set

$$y_{\text{pi}} = Ct\cos 3t + Dt\sin 3t.$$

Now, the C coefficient will turn out to be zero. To see this, consider what happens when two differentiations are made of $t\cos 3t$: we get a $t\cos 3t$ terms and a $\sin 3t$ term, neither of which balance with the $\cos 3t$ forcing term. I'll leave you to verify this by a direct substitution. So, just retaining the D term, substitution into the full equation gives,

$$[-9Dt\sin 3t + 6D\cos 3t] + 9Dt\sin 3t = \cos 3t.$$

The sine terms cancel and we get $D = \frac{1}{6}$. Hence the PI is

$$y_{\text{pi}} = \tfrac{1}{6}t\sin 3t,$$

and the general solution is

$$y = y_{\text{cf}} + y_{\text{pi}} = A\cos 3t + B\sin 3t + \tfrac{1}{6}t\sin 3t.$$

(b) The complementary function in this case is,

$$y_{\rm cf} = e^{-t}\Big(A\cos t + B\sin t\Big).$$

This featured as part of Question 1f.

(i) The Particular Integral is

$$y_{\rm pi} = \frac{e^{at}}{a^2+2a+2},$$

and so the general solution is

$$y = y_{\rm cf} + y_{\rm pi} = e^{-t}\Big(A\cos t + B\sin t\Big) + \frac{e^{at}}{a^2+2a+2}.$$

(ii) We set $y_{\rm pi} = Ct^2 + Dt + E$. Substitution into the ODE gives

$$[2C] + 2[2Ct + D] + 2[Ct^2 + Dt + E] = t^2.$$

Equating of like coefficients yields,

$$2C = 1, \qquad 4C + 2D = 0, \qquad 2C + 2D + 2E = 0.$$

Hence $C = \frac{1}{2}$, $D = -1$ and $E = \frac{1}{2}$. The PI is

$$y_{\rm pi} = \tfrac{1}{2}t^2 - t + \tfrac{1}{2}.$$

The general solution is

$$y = y_{\rm cf} + y_{\rm pi} = e^{-t}\Big(A\cos t + B\sin t\Big) + \tfrac{1}{2}t^2 - t + \tfrac{1}{2}.$$

(iii) We set $y_{\rm pi} = C\cos at + D\sin at$. We need both here because we have a first derivative in the ODE. Substitution into the ODE can get a bit messy, and it is probably best to keep the sines and cosines separate from the start. We get

$$\cos at\Big[-a^2C + 2aD + 2C\Big] + \sin at\Big[-a^2D - 2aC + 2D\Big] = \cos at.$$

The bracket multiplying the sines gives us

$$C = \frac{2-a^2}{2a}D. \qquad (1)$$

Equating the cosine coefficients gives,

$$(2-a^2)C + 2aD = 1, \qquad (2)$$

which, when we have substituted for C from above, gives

$$\Big[\frac{(2-a^2)^2}{2a} + 2a\Big]D = 1$$

Multiplication by $2a$ gives a tidier form,

$$\Big[(2-a^2)^2 + 4a^2\Big]D = 2a, \qquad \text{or} \qquad (a^4+4)D = 2a,$$

and hence

$$D = \frac{2a}{a^4+4}.$$

Therefore C is given by

$$C = \frac{2-a^2}{a^4+4}.$$

The general solution is

$$y = y_{\rm cf} + y_{\rm pi} = e^{-t}\Big(A\cos t + B\sin t\Big) + \frac{(2-a^2)\cos at + 2a\sin at}{a^4+4}.$$

Alternative route: It is also possible to use matrix methods to determine these coefficients. Equations (1) and (2) may be rearranged into the form,

$$\begin{pmatrix} 2-a^2 & 2a \\ -2a & 2-a^2 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence

$$\begin{aligned}
\begin{pmatrix} C \\ D \end{pmatrix} &= \begin{pmatrix} 2-a^2 & 2a \\ -2a & 2-a^2 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{(2-a^2)^2+4a^2}\begin{pmatrix} 2-a^2 & -2a \\ 2a & 2-a^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{a^4+4}\begin{pmatrix} 2-a^2 & -2a \\ 2a & 2-a^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{a^4+4}\begin{pmatrix} 2-a^2 \\ 2a \end{pmatrix}.
\end{aligned}$$

Hence C and D are given as above. [Should this stuff be new to you, then wait until the matrices section of the course before revisiting it.]

(c) The Complementary Function in this case is,

$$y_{\rm cf} = Ae^{3t} + Be^{4t}.$$

This means that all the forcing terms given in the question are standard cases except for (ii). The solutions are:

$$\begin{aligned}
y &= y_{\rm cf} + y_{\rm pi} = Ae^{3t} + Be^{4t} + \tfrac{1}{2}e^{2t} && (i)\\
y &= y_{\rm cf} + y_{\rm pi} = Ae^{3t} + Be^{4t} - te^{3t} && (ii)\\
y &= y_{\rm cf} + y_{\rm pi} = Ae^{3t} + Be^{4t} + \tfrac{1}{12}t^2 + \tfrac{14}{144}t + \tfrac{74}{1728} && (iii)\\
y &= y_{\rm cf} + y_{\rm pi} = Ae^{3t} + Be^{4t} + \frac{(12-a^2)\cos at - 7a\sin at}{a^4+25a^2+144} && (iv)
\end{aligned}$$

(d) This one is a special case with a vengeance. The Auxiliary Equation has three repeated roots, $\lambda = -1, -1, -1$, and therefore

$$y_{\rm cf} = (A + Bt + Ct^2)e^{-t}.$$

If we had just an e^{-t} as the forcing term, then we would have expected to use $y_{\rm pi} = Dt^3e^{-t}$ as the PI. In the present case we should use

$$y_{\rm pi} = (Dt^3 + Et^4 + Ft^5 + Gt^6)e^{-t}.$$

After loads of algebra we eventually get to,

$$y_{\rm pi} = \tfrac{1}{120}t^6e^{-t},$$

and hence the full solution of the original equation is

$$y = y_{\rm cf} + y_{\rm pi} = \Big(A + Bt + Ct^2 + \tfrac{1}{120}t^6\Big)e^{-t}.$$

- Now we use the given substitution, $y(t) = z(t)e^{-t}$. We get, in turn,

$$y' = (z' - z)e^{-t},$$

$$y'' = (z'' - 2z' + z)e^{-t},$$

$$y''' = (z''' - 3z'' + 3z' - z)e^{-t}.$$

Substitution of these expression into the governing equation results in a huge number of cancellations, and the surviving terms are,

$$\frac{d^3z}{dt^3} = t^3.$$

This is solved easily by integrating three times, not forgetting to introduce arbitrary constants each time, and we find that

$$z = A + Bt + Ct^2 + \tfrac{1}{120}t^6,$$

from which we then recover the full solution given above.

Q3. (a) Solve the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = e^{2t} \qquad y(0) = 0, \quad \left.\frac{dy}{dt}\right|_{t=0} = 0,$$

using standard CF/PI methods.

(b) Now we will attempt to solve the same equation using a slightly different method. Let us first find the solution of

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = e^{at} \qquad y(0) = 0, \quad \left.\frac{dy}{dt}\right|_{t=0} = 0,$$

where $a \neq 2$. Now let $a \to 2$ in the answer, and use L'Hôpital's rule to recover the solution when $a = 2$.

A3. If we use CF/PI methods, then we need to find the CF first. Setting $y_{\rm cf} = e^{\lambda t}$ in the homogeneous version of the equation,

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0,$$

we get

$$\lambda^2 + \lambda - 6 = 0.$$

The left hand side factorises into $(\lambda + 3)(\lambda - 2)$, and therefore $\lambda = 2, -3$. The CF is

$$y_{\rm cf} = Ae^{2t} + Be^{-3t},$$

where A and B are presently unknown.

(a) The forcing term, however, is of the same type as the first part of the CF and therefore we need to set $y_{\rm pi} = Cte^{2t}$ as the PI. Substitution into the ODE gives,

$$C\Big[(4t + 4)e^{2t} + (2t + 1)e^{2t} - 6te^{2t}\Big] = e^{2t}.$$

The left hand side simplifies, and we get,

$$5Ce^{2t} = e^{2t},$$

from which we find that $C = \frac{1}{5}$. Hence the general solution is

$$y = Ae^{2t} + Be^{-3t} + \tfrac{1}{5}te^{2t}.$$

Now we need to apply the initial conditions. At $t = 0$ we have $y = 0$, and therefore,

$$0 = A + B.$$

The second initial condition involves y', which is

$$y' = 2Ae^{2t} - 3Be^{-3t} + \tfrac{1}{5}(2t+1)e^{2t}.$$

As $y' = 0$ when $t = 0$, we get

$$0 = 2A - 3B + \tfrac{1}{5}.$$

On solving these two equations for A and B we get

$$A = -\tfrac{1}{25}, \qquad B = \tfrac{1}{25}.$$

Therefore the solution we seek is

$$y = -\tfrac{1}{25}e^{2t} + \tfrac{1}{25}e^{-3t} + \tfrac{1}{5}te^{2t}.$$

(b) When e^{at} is on the right hand side, the CF is the same as derived earlier with the arbitrary constants, and $y_{\rm pi} = Ce^{at}$ may be set whenever $a \neq 2$ and $a \neq -3$. It is straightforward to show that

$$y_{\rm pi} = \frac{e^{at}}{a^2 + a - 6}.$$

Therefore the general solution is

$$y = Ae^{2t} + Be^{-3t} + \frac{e^{at}}{a^2 + a - 6}.$$

The application of $y(0) = 0$ gives,

$$0 = A + B + \frac{1}{a^2 + a - 6}.$$

The application of $y'(0) = 0$ gives,

$$0 = 2A - 3B + \frac{a}{a^2 + a - 6}.$$

These equations have solutions,

$$A = -\frac{a+3}{5(a^2 + a - 6)}, \qquad B = \frac{a-2}{5(a^2 + a - 6)}.$$

The solution is therefore,

$$y = \frac{-(a+3)e^{2t} + (a-2)e^{-3t} + 5e^{at}}{5(a^2 + a - 6)}. \tag{1}$$

Now, if we substitute $a = 2$ directly into this solution, we get a zero-divide-zero problem, and therefore we need to use L'Hôpital's rule to do this properly.

Recall that L'Hôpital's rule in this context takes the form,

$$\lim_{a \longrightarrow 2} \frac{f(a)}{g(a)} = \frac{\left.\dfrac{df}{da}\right|_{a=2}}{\left.\dfrac{dg}{da}\right|_{a=2}},$$

provided that $f(2) = g(2) = 0$ and both of the derivatives are nonzero at $a = 2$. Therefore we have to differentiate the denominator and numerator with respect to a, regarding t as if it were a constant. Note, therefore, that the a-derivative of e^{at} is te^{at}. Therefore we get

$$y = \left.\frac{-e^{2t} + e^{-3t} + 5te^{at}}{5(2a+1)}\right|_{a=2} = \frac{-e^{-2t} + e^{3t} + 5te^{2t}}{25},$$

which is the same as in Q3a.

Q4. In this question the equation,

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = te^{-2t},$$

will be solved in two different ways.

(a) Use the Complementary Function/Particular Integral approach.

(b) Use the substitution $y = z(t)e^{-2t}$ to simplify the equation. You should then be able integrate the resulting equation once with respect to t. The final first order equation for z may then be solved using the CF/PI approach.

A4. (a) The Complementary Function is

$$y_{\rm cf} = Ae^{-2t} + Be^{-3t}.$$

If the forcing term in the ODE had been e^{-2t}, then we would have taken $y_{\rm pi} = Cte^{-2t}$. But as it is te^{-2t}, we need to take

$$y_{\rm pi} = (Ct + Dt^2)e^{-2t}.$$

Substituting this into the ODE gives,

$$C\Big[(4t-4) + 5(-2t+1) + 6t\Big]e^{-2t} + D\Big[(4t^2 - 8t + 2) + 5(-2t^2 + 2t) + 6t^2\Big]e^{-2t} = te^{-2t}.$$

Tidying this up and cancelling the exponentials both sides leads to

$$C + D(2t+2) = t.$$

Therefore $D = \frac{1}{2}$ and $C = -1$, and the full solution is

$$y = y_{\rm cf} + y_{\rm pi} = Ae^{-2t} + Be^{-3t} + (\tfrac{1}{2}t^2 - t)e^{-2t}.$$

(b) Given the substitution $y = z(t)e^{-2t}$, we have

$$y' = (z' - 2z)e^{-2t} \qquad \text{and} \qquad y'' = (z'' - 4z' + 4z)e^{-2t}.$$

Substitution into the ODE gives,

$$\Big[(z'' - 4z' + 4z)e^{-2t} + 5(z' - 2z)e^{-2t} + 6ze^{-2t}\Big] = te^{-2t}.$$

On tidying this up and cancelling the exponentials, we get

$$z'' + z' = t.$$

Integrating once gives,

$$z' + z = \tfrac{1}{2}t^2 + A.$$

This first order equation may be solved in one of two ways, as a first order linear using an integrating factor (forget it, there's integration by parts to do for this one!) or using CF/PI. The CF is $z_{\rm cf} = Be^{-t}$. The PI is

$$z_{\rm pi} = \tfrac{1}{2}t^2 - t + A + 1.$$

As A is arbitrary, we may redefine it slightly, and use

$$z_{\rm pi} = \tfrac{1}{2}t^2 - t + A.$$

The full solution in terms of z is

$$z = z_{\rm cf} + z_{\rm pi} = Be^{-t} + \tfrac{1}{2}t^2 - t + A.$$

Reverting back to y, we have

$$y = Be^{-3t} + (\tfrac{1}{2}t^2 - t)e^{-2t} + Ae^{-2t}.$$