Department of Mechanical Engineering, University of Bath

Mathematics ME10305

Problem Sheet 1 — ODE solutions

Q1. What is the order of the following equations or systems of equations?

In each case rewrite them in first order form.

Are they linear or nonlinear, and do they constitute Initial Value Problems or Boundary Value Problems? (Primes denote derivatives with respect to t.)

(a) $y'' + ty = 0$ subject to $y(0) = 0, \quad y'(0) = 1.$

(b) $y''' + y'' - 2yz = 0, \quad z' = ty$ subject to $y(0) = 1, \quad y'(0) = 0, \quad y'(\infty) = 0, \quad z(0) = 0.$

(c) $y'''' + 2(y + y'')^3 y' + y^5 = 1,$ subject to $y = y' = y'' = y''' - 1 = 0$ at $t = 0$.

(d) $x'' + 2x - y = 0, \quad y'' - x + 2y - z = 0, \quad z'' - 3y + 2z = 0,$
subject to $x(0) = 1, \quad x'(0) = 0, \quad y(0) = y'(0) = 0, \quad z(0) = z'(0) = 0.$

(e) $f' = g, \quad g'' + fg' + f'g = 0,$ subject to $f(0) = 0, \quad g(0) = 1, \quad g(\infty) = 0.$

A1. In all cases we shall use subscripts to denote the different variables which are needed when reducing the equations/systems to first order form. This is to mimic what will be required when solving such systems numerically.

(a) This is a linear, 2nd order, initial value problem. We therefore require two new variables in order to reduce it to first order form: $y_1 = y$, $y_2 = y'$. Hence the first order form is,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -ty_1 \end{aligned} \qquad \text{subject to} \qquad \begin{aligned} y_1(0) &= 0 \\ y_2(0) &= 1 \end{aligned}.$$

(b) The y-equation is of 3rd order, while the z-equations is of 1st order. Hence the combined system is of 4th order, and therefore we shall need 4 variables for reduction to first order form, 3 for y and 1 for z. The system is nonlinear as there is a yz term. It is also a boundary value problem as boundary conditions are given at both $t = 0$ and as $t \to \infty$.

$$\text{Setting} \quad \begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ y_4 &= z \end{aligned} \quad \Rightarrow \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= 2y_1y_4 - y_3 \\ y_4' &= ty_1 \end{aligned} \quad \text{subject to} \quad \begin{array}{ll} y_1(0) = 1 & \\ y_2(0) = 0 & y_2(\infty) = 0 \\ & \\ y_4(0) = 0 & \end{array}$$

(c) This equation is of 4th order as the highest derivative is a fourth derivative; don't be put off by the y^5, as this is only a fifth power. The equation is nonlinear. It is also an Initial Value Problem as all the boundary conditions are given at $t = 0$. (I have written them in a slightly unusual way on the question sheet; the more usual manner would be to write $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ and $y'''(0) = 1$.)

$$\text{Setting} \quad \begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ y_4 &= y''' \end{aligned} \quad \Rightarrow \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= 1 - 2y_2(y_1 + y_3)^3 - y_1^5 \end{aligned} \quad \text{subject to} \quad \begin{aligned} y_1(0) &= 0 \\ y_2(0) &= 0 \\ y_3(0) &= 0 \\ y_4(0) &= 1 \end{aligned}$$

(d) This is a 6th order system as it is composed of three 2nd order equations. It is also a linear, initial value problem.

$$\text{Setting}\quad \begin{aligned} y_1 &= x\\ y_2 &= x'\\ y_3 &= y\\ y_4 &= y'\\ y_5 &= z\\ y_6 &= z' \end{aligned} \quad\Rightarrow\quad \begin{aligned} y_1' &= y_2\\ y_2' &= y_3 - 2y_1\\ y_3' &= y_4\\ y_4' &= y_1 - 2y_3 + y_5\\ y_5' &= y_6\\ y_6' &= 3y_3 - 2y_5 \end{aligned} \quad\text{subject to}\quad \begin{aligned} y_1(0) &= 1\\ y_2(0) &= 0\\ y_3(0) &= 0\\ y_4(0) &= 0\\ y_5(0) &= 0\\ y_6(0) &= 0 \end{aligned}$$

(e) This is a third order, nonlinear, boundary value problem.

$$\text{Setting}\quad \begin{aligned} y_1 &= f\\ y_2 &= g\\ y_3 &= g' \end{aligned} \quad\Rightarrow\quad \begin{aligned} y_1' &= y_2\\ y_2' &= y_3\\ y_3' &= -y_1y_3 - y_2^2 \end{aligned} \quad\text{subject to}\quad \begin{aligned} y_1(0) &= 0\\ y_2(0) &= 1 \quad y_2(\infty) = 0 \end{aligned}$$

Q2. Solve the following equations by direct integration of both sides.
(a) $y' = \cos t$ subject to $y(0) = 1$, (b) $y' = e^{2t} + 1$ subject to $y(1) = 1$.

A2. These are the easiest type of equations to solve since they involve only integration, and therefore they could be regarded as being integrals in disguise. Technically they are also of variables-separable type ($y' = f(y)g(t)$), but the y-dependent function is absent on the right hand side.

(a) Integrating once yields $y = \sin t + c$. Application of the initial condition gives $1 = \sin 0 + c$, from which we get $c = 1$. Hence the final solution is $y = 1 + \sin t$.

(b) Using the same procedure we get $y = \frac{1}{2}e^{2t} + t + c$. The initial condition gives $1 = \frac{1}{2}e^2 + 1 + c$. Therefore the final solution is

$$y = \tfrac{1}{2}[e^{2t} - e^2] + t.$$

Note that there is also an integral form for the solution of $y' = f(t)$ subject to $y(a) = b$; it is

$$y = b + \int_a^t f(\tau)\, d\tau.$$

Notice how the lower limit of the integral allows the correct initial condition to be applied easily.

Application of this formula for part (a) yields,

$$y = 1 + \int_0^t \cos\tau\, d\tau = 1 + \Big[\sin\tau\Big]_0^t = 1 + \sin t.$$

For part (b) we have

$$y = 1 + \int_1^t \Big[e^{2\tau} + 1\Big]\, d\tau = 1 + \Big[\tfrac{1}{2}e^{2\tau} + \tau\Big]_1^t = 1 + \tfrac{1}{2}\Big[e^{2t} - e^2\Big] + \Big[t - 1\Big] = \tfrac{1}{2}\Big[e^{2t} - e^2\Big] + t.$$

Q3. Use separation of variables to find the solutions to the following ODEs,

(a) $\dfrac{dy}{dt} = \dfrac{4t}{y}$, $y(0) = 1$, (b) $\dfrac{dy}{dt} = 3t^2y$, $y(0) = 1$,

(c) $\dfrac{dy}{dt} = t(1 + y^2)$ $y(0) = 1$, (d) $\dfrac{dy}{dt} = t^2(1 - y^2)$, $y(0) = 2$,

(e) $\dfrac{dy}{dt} = y - y^2$, $y(0) = 2$, (f) $t^2\dfrac{dy}{dt} = y - t^3y$, $y(1) = 1$,

(g) $\dfrac{d^2y}{dt^2} = \dfrac{1}{t}\dfrac{dy}{dt}$, $y(0) = 1$, $y'(1) = 2$.

A3. In each case we move all the y-dependant terms to the left and all the t-dependent terms to the right.

(a) $\dfrac{dy}{dt} = \dfrac{4t}{y} \quad\Longrightarrow\quad y\,dy = 4t\,dt \quad\Longrightarrow\quad \tfrac{1}{2}y^2 = 2t^2 + c \quad\Longrightarrow\quad y = \sqrt{4t^2 + 2c}.$

Note that we could easily redefine the arbitrary constant to obtain an easier–looking expression: $y = \sqrt{4t^2 + c}$.

If we apply the condition, $y(0) = 1$, then the 'easier-looking' expression gives, $1 = \sqrt{c}$, and hence $c = 1$. The solution is $y = \sqrt{4t^2 + 1}$. Of course, if we had used the original expression for the solution, $y = \sqrt{4t^2 + 2c}$ then the boundary condition leads to $c = \frac{1}{2}$, and thence to $y = \sqrt{4t^2 + 1}$. The moral of the story is that the solution which is obtained after using the boundary conditions will be independent of the way in which one writes down the arbitrary constants.

(b) $\dfrac{dy}{dt} = 3t^2 y \quad\Longrightarrow\quad \dfrac{dy}{y} = 3t^2\,dt \quad\Longrightarrow\quad \ln y = t^3 + c \quad\Longrightarrow\quad y = e^{t^3 + c}.$

Again we may write this in the easier form: $y = Ae^{t^3}$ where $A = e^c$ is a new arbitrary constant.

Application of the condition, $y(0) = 1$, leads either to $c = 0$ or $A = 1$ depending on which version of the general solution is used. The final solution is $y = e^{t^3}$.

(c) $\dfrac{dy}{dt} = t(1 + y^2) \quad\Longrightarrow\quad \dfrac{dy}{1 + y^2} = t\,dt \quad\Longrightarrow\quad \tan^{-1} y = \tfrac{1}{2}t^2 + c \quad\Longrightarrow\quad y = \tan(\tfrac{1}{2}t^2 + c).$

Here I have used a standard result to get the $\tan^{-1}$ answer.

The boundary condition, $y(0) = 1$ yields $1 = \tan(c)$. There are multiple possible values for c ($\pi/4 + n\pi$, for integer values of n), but all will yield the same shape of graph when drawn. Therefore we will choose to use $c = \pi/4$. Hence $y = \tan(\frac{1}{2}t^2 + \pi/4)$.

(d) $\dfrac{dy}{dt} = t^2(1 - y^2) \quad\Longrightarrow\quad \dfrac{dy}{1 - y^2} = t^2\,dt.$

Further progress is made by using partial fractions to simplify the left hand side; we have

$$\frac{1}{1 - y^2} = \tfrac{1}{2}\Big[\frac{1}{y+1} - \frac{1}{y-1}\Big].$$

Therefore we get

$$\tfrac{1}{2}\Big[\frac{1}{y+1} - \frac{1}{y-1}\Big]\,dy = t^2\,dt.$$

Integrating both sides gives

$$\tfrac{1}{2}\ln\Big|\frac{y+1}{y-1}\Big| = \tfrac{1}{3}t^3 + c.$$

On moving the $\frac{1}{2}$ to the right hand side, and setting exp[LHS]=exp[RHS], we get

$$\Big|\frac{y+1}{y-1}\Big| = e^{2t^3/3 + 2c} = e^{2c}e^{2t^3/3} = Ae^{2t^3/3}.$$

Here we have defined $A = e^{2c}$ as the new arbitrary constant. Moreover, as A is arbitrary, we may drop the moduli from the left hand side:

$$\frac{y+1}{y-1} = Ae^{2t^3/3}.$$

Rearranging for y gives,

$$y = \frac{Ae^{2t^3/3} + 1}{Ae^{2t^3/3} - 1}.$$

Note that it is very possible to have quite a few other ways of writing this solution down. Division of both denominator and numerator by $Ae^{2t^3/3}$ and setting $B = 1/A$, gives the alternative,

$$y = \frac{1 + Be^{-2t^3/3}}{1 - B^{-2t^3/3}},$$

which, in many ways, is clearer than the one we found first.

Application of $y(0)=2$ gives $A=3$ or $B=\frac{1}{3}$. Hence,

$$y=\frac{1+3e^{2t^3/3}}{3e^{2t^3/3}-1} \qquad \text{or} \qquad y=\frac{1+\frac{1}{3}e^{-2t^3/3}}{1-\frac{1}{3}e^{-2t^3/3}}.$$

(e) $\dfrac{dy}{dt}=y-y^2 \quad \Longrightarrow \quad \dfrac{dy}{y-y^2}=dt \quad \Longrightarrow \quad \left[\dfrac{1}{y}-\dfrac{1}{y-1}\right]dy=dt.$

From this we get

$$\ln\left|\frac{y}{y-1}\right|=t+c.$$

Using similar ideas to the previous question eventually gives

$$y=\frac{Ae^t}{Ae^t+1} \qquad \text{or, equivalently,} \qquad y=\frac{1}{1+Be^{-t}}.$$

The application of the initial condition gives,

$$y=\frac{1}{1-\frac{1}{2}e^{-t}}.$$

(f) $t^2\dfrac{dy}{dt}=y-t^3y \quad \Longrightarrow \quad \dfrac{dy}{y}=(t^{-2}-t)\,dt \quad \Longrightarrow \quad \ln|y|=-(t^{-1}+t^2/2)+c \quad \Longrightarrow \quad y=A\exp\Big[-\Big(\dfrac{1}{t}+\dfrac{t^2}{2}\Big)\Big].$

The boundary condition gives $A=e^{3/2}$ and hence,

$$y=\exp\Big[\frac{3}{2}-\frac{1}{t}-\frac{t^2}{2}\Big].$$

(g) $\dfrac{d^2y}{dt^2}=\dfrac{1}{t}\dfrac{dy}{dt}$. This is a first order equation in y'. So let $v=y'$ to get

$$\frac{dv}{dt}=\frac{v}{t} \quad \Longrightarrow \quad \frac{dv}{v}=\frac{dt}{t} \quad \Longrightarrow \quad \ln v=\ln t+c \quad \Longrightarrow \quad v=At \quad \Longrightarrow \quad \frac{dy}{dt}=At \quad \Longrightarrow \quad y=\tfrac{1}{2}At^2+B.$$

The initial conditions eventually yield,

$$y=1+t^2.$$

Q4. Find the Integrating Factor and hence solve the following 1st order equations.

(a) $\dfrac{dy}{dt}+\dfrac{y}{t}=1$, (b) $\dfrac{dy}{dt}-\dfrac{y}{t}=1$, (c) $\dfrac{dy}{dt}+\dfrac{3y}{t}=t^{-2}$, (d) $\dfrac{dy}{dt}+2ty=2t$,

(e) $\dfrac{dy}{dt}+y\cot t=1$, (f) $\dfrac{dy}{dt}+\dfrac{1+2t}{t}y=\dfrac{1}{t}$, (g) $\dfrac{dy}{dt}+4t^3y=t^3$ (h) $t\dfrac{dy}{dt}+(t+1)y=t^2$.

A4.

(a) $\dfrac{dy}{dt}+\dfrac{y}{t}=1.$ Hence the integrating factor is $e^{\int t^{-1}\,dt}=e^{\ln t}=t.$

On multiplying the equation by the integrating factor we get

$$t\frac{dy}{dt}+y=t \quad \Longrightarrow \quad \frac{d(ty)}{dt}=t \quad \Longrightarrow \quad ty=\tfrac{1}{2}t^2+c \quad \Longrightarrow \quad y=\tfrac{1}{2}t+ct^{-1}.$$

(b) This one is almost the same as the previous equation, the only difference being the minus sign which is very important. We have

$$\frac{dy}{dt} - \frac{y}{t} = 1. \qquad \text{Hence the integrating factor is } e^{-\int t^{-1}\,dt} = e^{-\ln t} = t^{-1}.$$

On multiplying the equation by the integrating factor we get

$$\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y = \frac{1}{t} \quad\Longrightarrow\quad \frac{dy/t)}{dt} = t^{-1} \quad\Longrightarrow\quad \frac{y}{t} = c + \ln|t| \quad\Longrightarrow\quad y = ct + t\ln|t|.$$

So clearly the different sign results in a very very different solution. Note that the y in the given ODE is multiplied by $-1/t$, and it is this which needs to be used in the formula for the Integrating Factor.

(c) $\dfrac{dy}{dt} + \dfrac{3y}{t} = t^{-2}$. Hence the integrating factor is $e^{\int 3t^{-1}\,dt} = e^{3\ln t} = t^3$.

The equation becomes

$$t^3\frac{dy}{dt} + 3t^2y = t \quad\Longrightarrow\quad \frac{d(t^3y)}{dt} = t \quad\Longrightarrow\quad t^3y = \tfrac{1}{2}t^2 + c \quad\Longrightarrow\quad y = \tfrac{1}{2}t^{-1} + ct^{-3}.$$

(d) $\dfrac{dy}{dt} + 2ty = 2t$. Hence the integrating factor is $e^{\int 2t\,dt} = e^{t^2}$.

We get

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 2te^{t^2} \quad\Longrightarrow\quad \frac{d(e^{t^2}y)}{dt} = 2te^{t^2}\left[= \frac{d[e^{t^2}]}{dt}\right] \quad\Longrightarrow\quad e^{t^2}y = e^{t^2} + c \quad\Longrightarrow\quad y = 1 + ce^{-t^2}.$$

(e) $\dfrac{dy}{dt} + \cot t\, y = 1$. Hence the integrating factor is $e^{\int \cot t\,dt} = e^{\int \frac{\cos t}{\sin t}dt} = e^{\ln\sin t} = \sin t$.

The ODE becomes,

$$\sin t\frac{dy}{dt} + (\cos t)y = \sin t \quad\Longrightarrow\quad (y\sin t)' = \sin t \quad\Longrightarrow\quad y\sin t = -\cos t + c \quad\Longrightarrow\quad y = -\cot t + c/\sin t.$$

(f) $\dfrac{dy}{dt} + \dfrac{1+2t}{t} = \dfrac{1}{t}$. Hence the integrating factor is $e^{\int (2+t^{-1})\,dt} = e^{2t+\ln t} = te^{2t}$.

We now get,

$$te^{2t}y' + (1+2t)e^{2t}y = e^{2t} \quad\Longrightarrow\quad (te^{2t}y)' = e^{2t} \quad\Longrightarrow\quad te^{2t}y = \tfrac{1}{2}e^{2t} + c \quad\Longrightarrow\quad y = \tfrac{1}{2}t^{-1}(1 + ce^{-2t}),$$

where I have redefined the value of c, the arbitrary constant.

(g) $\dfrac{dy}{dt} + 4t^3y = t^3$. Hence the integrating factor is e^{t^4}

Therefore we have,

$$(e^{t^4}y)' = t^3e^{t^4} \quad\Longrightarrow\quad e^{t^4}y = \tfrac{1}{4}e^{t^4} + c \quad\Longrightarrow\quad y = \tfrac{1}{4} + ce^{-t^4}.$$

(h) The given ODE is $t\dfrac{dy}{dt} + (t+1)y = t^2$, but it isn't in the standard form. The derivative term must have a unit coefficient. Therefore we first divide by t:

$$\frac{dy}{dt} + \frac{(t+1)}{t}y = t.$$

The integrating factor is found by first noting that $(t+1)/t = 1 + t^{-1}$. Hence

$$\text{I.F.} = e^{\int (1+t^{-1})\,dt} = e^{t+\ln|t|} = te^t.$$

Multiplying by this yields,

$$te^t\frac{dy}{dt} + (t+1)e^t\,y = t^2e^t,$$

or, on collecting terms,

$$\frac{dyte^t}{dt} + (t+1)e^t\,y = t^2e^t.$$

We need to integrate once with respect to t, and integration of the right hand side will need integration by parts. We get,

$$yte^t = (t^2-2t+2)e^t + c.$$

Hence

$$y = \frac{(t^2-2t+2)}{t} + \frac{ce^{-t}}{t}.$$

Q5. The following equation

$$\frac{dy}{dt} = y^3 - y$$

falls into two different categories. First, it is of variables-separable type, and second it is an example of what is known as a Bernoulli equation.

(i) Use separation of variables, followed by partial fractions to find the solution subject to the initial condition that $y = 1/\sqrt{2}$ when $t = 0$.

(ii) Solve the equation again by first using the substitution, $y = z^{-1/2}$ where $z = z(t)$ is a new dependent variable. This substitution should give you a linear equation for z.

A5.

(i) Separation of variables results in

$$\frac{dy}{y^3-y} = dt.$$

Using partial fractions to simplify the left hand side yields

$$\left[\frac{1}{y+1} - \frac{2}{y} + \frac{1}{y-1}\right]dy = 2\,dt.$$

Integration gives,

$$\ln|y+1| - 2\ln|y| + \ln|y-1| = 2t + c.$$

Simplification of the left hand side gives,

$$\ln\left|\frac{(y+1)(y-1)}{y^2}\right| = \ln\left|\frac{y^2-1}{y^2}\right| = 2t + c.$$

Hence

$$\frac{y^2-1}{y^2} = e^{2t+c} = Ae^{2t}.$$

Solving for y^2 gives,

$$y^2 = \frac{1}{1-Ae^{2t}},$$

and so

$$y = \pm\sqrt{\frac{1}{1-Ae^{2t}}}. \qquad (*)$$

The application of the initial condition should give a value for A and resolve which of the two signs should be used. As $y(0) = 1/\sqrt{2}$, we have

$$\frac{1}{\sqrt{2}} = \pm\sqrt{\frac{1}{1-A}}.$$

Therefore we should take the plus sign and set $A = -1$. The solution is

$$y = \sqrt{\frac{1}{1+e^{2t}}}.$$

(ii) We are guided to use the substitution, $y = z^{-1/2}$, in order to change the dependent variable from y to z. Given that the function of a function rule states,

$$\frac{dy}{dt} = \frac{dy}{dz}\frac{dz}{dt},$$

then

$$\frac{dy}{dt} = -\frac{1}{2z^{3/2}}\frac{dz}{dt}.$$

This and the definition of z are substituted into the ODE to give

$$-\frac{1}{2z^{3/2}}\frac{dz}{dt} = \frac{1}{z^{3/2}} - \frac{1}{z^{1/2}}.$$

Multiplication by $-2z^{3/2}$ simplifies the equation greatly:

$$\frac{dz}{dt} - 2z = -2.$$

This is a first-order linear equation and it may be solved using the Integrating Factor approach [it is also has constant coefficients, and therefore it is also amenable to the Particular Integral/Complementary Function method detailed in lectures 3 and 4]. The integrating factor is e^{-2t}, and we get

$$e^{-2t}\frac{dz}{dt} - 2e^{-2t}z = -2e^{-2t}.$$

The left hand side is an exact differential and so

$$\frac{d}{dt}\Big(e^{-2t}z\Big) = -2e^{-2t}.$$

Integration gives,

$$e^{-2t}z = e^{-2t} + c \quad\Longrightarrow\quad z = 1 + ce^{2t}.$$

Returning to the y-variable, we have

$$y = \pm\frac{1}{\sqrt{1+ce^{2t}}},$$

which is essentially the same as the starred equation above. The solution, after using the initial condition, is identical.

Q6. The general form for Bernoulli's equation is

$$\frac{dy}{dt} + P(t)y^n + Q(t)y = 0,$$

where $P(t)$ and $Q(t)$ are given functions.

(i) Use the substitution $y = z^\alpha$, where z is a function of time and where α is constant to be found. After substitution, determine that value of α which reduces the equation for z into one of first order linear form.

(ii) You are now required to solve the ODE,

$$\frac{dy}{dt} + y - y^{-1} = 0, \qquad \text{subject to} \qquad y = 2 \text{ at } t = 0.$$

Use your general Bernoulli result to reduce the ODE to first order linear form and solve it. Check your answer for y by substituting it back into the above equation.

(iii) The above equation may also be solved using separation of variables. Please do it this way as well.

A6.

(i) If we set $y = z^{\alpha}$, then $\dfrac{dy}{dt} = \alpha z^{\alpha-1}\dfrac{dz}{dt}$, and therefore the ODE becomes,

$$\alpha z^{\alpha-1}\frac{dz}{dt} + P(t)z^{n\alpha} + Q(t)z^{\alpha} = 0.$$

On multiplying throughout by $z^{1-\alpha}$ — be careful with the exponents! — we get,

$$\alpha\frac{dz}{dt} + P(t)z^{n\alpha+1-\alpha} + Q(t)z = 0.$$

This equation will be of first order linear form if and only if $P(t)$ is multiplied by a constant. Therefore the exponent of z must be zero. Hence

$$n\alpha + 1 - \alpha = 0 \qquad \Rightarrow \qquad \alpha = \frac{1}{1-n}.$$

We may now check the substitution used in Q5, where $n = 3$. This value yields $\alpha = -1/2$, as given.

(ii) For the equation, $y' + y - y^{-1} = 0$, we have $n = -1$ and therefore $\alpha = 1/2$ using the above analysis. The substitution to be used is $y = z^{1/2}$. Eliminating all details, the ODE for z becomes,

$$\frac{dz}{dt} + 2z = 2,$$

which has general solution,

$$z = 1 + Ae^{-2t}.$$

The initial condition is that $y(0) = 2$, and therefore $z(0) = 4$ since $y = z^{1/2}$. Hence $A = 3$ and therefore we have

$$z = 1 + 3e^{-2t}.$$

In terms of y we have,

$$y = \sqrt{1 + 3e^{-2t}}.$$

Finally, note that the general solution for z, namely, $z = 1 + Ae^{-2t}$ could also have been converted to a general solution for y, namely, $y = \sqrt{1 + Ae^{-2t}}$, and then the given initial condition could have been applied to obtain the same solution.

(iii) It is possible to rearrange the given equation into the form,

$$\frac{dy}{dt} = y^{-1} - y = \frac{1-y^2}{y}.$$

This may be rearranged as

$$\frac{y}{y^2-1}\,dy = -dt,$$

where I have also multiplied both sides by -1 to get a $+y^2$ in the denominator. Multiplying by two and 'separating the variables' yields

$$\frac{2y}{y^2-1}\,dy = -2\,dt,$$

and now we see that the left hand side is of the form, f'/f, for which the integral is a natural logarithm. Now integrate both sides to get,

$$\int \frac{2y}{y^2-1}\,dy = \int -2\,dt \qquad \Rightarrow \qquad \ln|y^2-1| = -2t + c.$$

Exponentiating both sides and using the usual trick of removing the modulus signs by replacing $\pm e^c$ by A, we get

$$y^2 - 1 = Ae^{-2t} \qquad \Rightarrow \qquad y^2 = 1 + Ae^{-2t}.$$

The initial condition is that $y(0) = 2$, and hence $A = 3$. Therefore we have the final solution,

$$y = \sqrt{1 + 3e^{-2t}},$$

where the positive square root was taken in order to comply with the given initial condition.

Q7. Another category of ODE could be called equidimensional. This is an example:

$$\frac{dy}{dx} = \frac{2y^2 + x^2}{2xy}.$$

The method of solution is to substitute $y(x) = x\,v(x)$ to form an ODE for $v(x)$. The resulting equation should then be solvable using separation of variables. Solve the above equation subject to the initial condition, $y = 1$ when $x = 1$. Then check that your solution satisfies the original ODE. [You may also attempt Q4a using the same idea.]

A7. On taking the derivative of $y = xv(x)$, we get,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

After substituting $y = xv$ into the given equation, we have

$$x\frac{dv}{dx} + v = \frac{2x^2v^2 + x^2}{2x^2v} \quad = \quad \frac{2v^2 + 1}{2v} \quad = \quad v + \frac{1}{2v}.$$

This simplifies to,

$$x\frac{dv}{dx} = \frac{1}{2v},$$

and so we can separate variables to give,

$$2v\,dv = \frac{dx}{x}.$$

Integration yields,

$$v^2 = \ln|x| + c \qquad \Rightarrow \qquad v = \sqrt{\ln|x| + c},$$

and hence

$$y = x\sqrt{c + \ln|x|}.$$

The initial condition, $y(1) = 1$, means that the arbitrary constant is $c = 1$. In addition, and given the initial condition is given at a positive value of x, we may remove the modulus signs in our solution. Therefore the final solution is,

$$y = x\sqrt{1 + \ln x} \qquad \text{for } x > 0.$$

- The addendum to this question, namely the application of the substitution to Q4a, may be done in the same way. If we set $y = v(t)t$ in $y' + y/t = 1$, then we get

$$t\frac{dv}{dt} + 2v = 1,$$

which may be rearranged to get,

$$\frac{dv}{dt} = \frac{1 - 2v}{t},$$

which is of variables-separable type. Therefore we separate the variables:

$$\frac{dv}{1 - 2v} = \frac{dt}{t}.$$

I don't like denominators of the type given on the left hand side (i.e. the -2 multiplying the v; I prefer 1 because it is safer!), and therefore I prefer to change this latest line to

$$\frac{dv}{v - \frac{1}{2}} = -2\frac{dt}{t}.$$

Integration gives, $\ln|v-\frac{1}{2}| = -2\ln|t| + c = -\ln t^2 + c.$

Exponentiating both sides gives,

$$v - \tfrac{1}{2} = At^{-2} \qquad \text{where} \qquad A = e^c,$$

and so

$$v = \tfrac{1}{2} + At^{-2}.$$

Therefore the final answer is, $y = \frac{1}{2}t + At^{-1},$

as before.
