University of Bath, Department of Mechanical Engineering

ME10305 Mathematics 2.

Matrices Sheet 3 — Eigenvalues, eigenvectors and solutions of ODEs systems.

Q1. Find the eigenvalues and eigenvectors of the following matrices;

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad F = \begin{pmatrix} b & a & 0 \\ c & b & a \\ 0 & c & b \end{pmatrix}, \quad G = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}.$$

ANSWER:

In general, when we need to find the eigenvalues of a matrix, M, we set $\det(M - \lambda I) = 0$.

(a) We set

$$0 = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1.$$

Hence $(2-\lambda)^2 = 1$, and so $\lambda = 1, 3$.

To find the eigenvalues we solve

$$\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and hence both equations yields $x = y$. Therefore the eigenvector is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and hence both equations yields $x = -y$. Therefore the eigenvector is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In the above, α and β are arbitrary. If we wanted to normalise the eigenvectors so that the largest amplitude entry is 1, then we could choose $\alpha = \beta = 1$. On the other hand, if we wished to make the length of each vector equal to 1, then $\alpha = \beta = 1/\sqrt{2}$ would be fine.

(b) This matrix is almost the same as the previous one. The eigenvalues are -2 and -4, and the corresponding eigenvectors are again $(1, 1)$ and $(1, -1)$.

(c) We'll set about this one the standard way, namely by setting $\det(C - \lambda I) = 0$:

$$\begin{aligned}
0 &= \begin{vmatrix} 1-\lambda & 3 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\Big[\lambda^2 - 5\lambda + 5\Big] + 1\Big[-3 + (4-\lambda)\Big] \qquad \text{1st column expansion} \\
&= (1-\lambda)\Big[\lambda^2 - 5\lambda + 5\Big] + (1-\lambda) \\
&= (1-\lambda)\Big[\lambda^2 - 5\lambda + 6\Big] \\
&= (1-\lambda)(2-\lambda)(3-\lambda).
\end{aligned}$$

Hence the three eigenvalues are $\lambda = 1, 2, 3$ — it looks as though I might have planned this!

When $\lambda = 1$, the equation to solve for the eigenvector is,

$$\begin{pmatrix} 0 & 3 & -1 \\ 0 & 3 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first row is equivalent to $3y - z = 0$, so if we set $y = \alpha$, then $z = 3\alpha$. The third row is equivalent to $x + y = 0$ and so $x = -\alpha$. Therefore the eigenvector corresponding to $\lambda = 1$ is,

$$\alpha \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}.$$

When $\lambda = 2$, we have

$$\begin{pmatrix} -1 & 3 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second row yields $2y - z = 0$, and therefore, if we set $y = \alpha$, then $z = 2\alpha$. Both rows 1 and 3 will now yield $x = \alpha$, and hence the eigenvector corresponding to $\lambda = 2$ is

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

When $\lambda = 3$, we have

$$\begin{pmatrix} -2 & 3 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second row yields $y - z = 0$, and therefore, if we set $y = \alpha$, then $z = \alpha$. Both rows 1 and 3 will now yield $x = \alpha$, and hence the eigenvector corresponding to $\lambda = 3$ is

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

If we wish to normalise these three eigenvectors, i.e. to make them have unit length, then they are

$$\frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \qquad \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(d) This 3×3 matrix has 2 down the main diagonal, and therefore it would be best to retain the $(2-\lambda)$ factors as long as possible, rather than multiplying out the cubic that is obtained. We get

$$
\begin{aligned}
0 &= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} \\
&= (2-\lambda)\Big[(2-\lambda)^2 - 1\Big] - 1\Big[(2-\lambda) - 1\Big] + 1\Big[1 - (2-\lambda)\Big] \\
&= (2-\lambda)\Big[(2-\lambda)^2 - 3\Big] + 2 \\
&= (2-\lambda)^3 - 3(2-\lambda) + 2, \\
&= \Big[(2-\lambda) - 1\Big]\,\Big[(2-\lambda)^2 + (2-\lambda) - 2\Big] \qquad \text{as } (2-\lambda) = 1 \text{ is a root} \\
&= \Big[1-\lambda\Big]\,\Big[\lambda^2 - 5\lambda + 4\Big] \\
&= (1-\lambda)(\lambda-1)(\lambda-4).
\end{aligned}
$$

Hence $\lambda = 1, 1, 4$. Now this gives an interesting situation, namely a matrix with a repeated eigenvalue. We'll deal with the other one first, though.

When $\lambda = 4$ we have,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we set $x = \alpha$ in the first equation, then

$$y + z = 2\alpha.$$

The second equation now becomes,

$$-2y + z = -\alpha.$$

Hence $y = z = \alpha$. The eigenvector is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

When $\lambda = 1$ we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Equation row of this matrix is equivalent to the equation for a plane. So let us set $y = \beta$ and $z = \gamma$, then $x = -\beta - \gamma$. Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\beta - \gamma \\ \beta \\ \gamma \end{pmatrix} = \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore we have two eigenvectors corresponding to the repeated eigenvalue. But note that these are not the only way in which this solution can be written; we could also have,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

and there are many others.

(e) The eigenvalues of this matrix are easy to find as the matrix is upper-triangular.

$$0 = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$
$$= (1-\lambda)(2-\lambda)(3-\lambda),$$

Hence $\lambda = 1, 2, 3$.

For $\lambda = 1$ we have,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The third row gives $z = 0$. The first two rows give $y = 0$. Therefore x may take any value. So we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 2$ we have,

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Rows 2 and 3 both yield $z = 0$. The first row gives $x = y$. Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda = 3$ we have,

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If, in row 2, we choose $z = \gamma$, then $y = 2\gamma$. Row 1 now gives $x = \frac{3}{2}\gamma$. Therefore,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \gamma \begin{pmatrix} 3/2 \\ 2 \\ 1 \end{pmatrix}.$$

(f) We will assume that a, b and c are real numbers. It will turn out eventually that we will need $ac > 0$ to have real eigenvalues and eigenvectors, but that will come about in due course. Proceeding in the usual way we have,

$$0 = \begin{vmatrix} b-\lambda & a & 0 \\ c & b-\lambda & a \\ 0 & c & b-\lambda \end{vmatrix} = (b-\lambda)\Big[(b-\lambda)^2 - ac\Big] - ac(b-\lambda) = (b-\lambda)\Big[(b-\lambda)^2 - 2ac\Big].$$

This needs to be zero, and hence

$$b - \lambda = 0, \ \pm\sqrt{2ac}.$$

So we get,

$$\lambda = b, \ b \pm \sqrt{2ac}.$$

It is at this point we need $ac > 0$ to ensure that the eigenvalues are real; it is possible to have complex eigenvalues and eigenvectors, but this is beyond the scope of the unit.

We get the following eigenvalue/eigenvector pairs:

$$\lambda = b: \quad \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix}; \qquad \lambda = b + \sqrt{2ac}: \quad \begin{pmatrix} a \\ \sqrt{2ac} \\ c \end{pmatrix}; \qquad \lambda = b - \sqrt{2ac}: \quad \begin{pmatrix} a \\ -\sqrt{2ac} \\ c \end{pmatrix}.$$

(g) Dead simple, this one. The $\det(G - \lambda I) = 0$ yields,

$$(a - \lambda)(b - \lambda)(c - \lambda) = 0,$$

and therefore $\lambda = a, b, c$. The respective eigenvectors are,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(h) This appears to be a diagonal matrix, but this isn't termed a diagonal matrix since the nonzero entries are not on the main diagonal. We get,

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + \lambda - 1 = -(\lambda + 1)(\lambda - 1)^2.$$

Therefore the three eigenvalues are, $\lambda = -1, 1, 1$, i.e. another case with a repeated eigenvalue.

Now we solve $(H - \lambda I)\underline{x} = 0$. When $\lambda = -1$ we have,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second equation gives $y = 0$. Hence the eigenvector is, $(1, 0, -1)^T$.

When $\lambda = 1$ we have,

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

There are two possible 'nice' eigensolutions, namely,

$$(1, 0, 1)^T \qquad \text{and} \qquad (0, 1, 0)^T,$$

but any two different sums of these two also work, such as

$$(1, 1, 1)^T \qquad \text{and} \qquad (1, -1, 1)^T.$$

(j) This matrix yields,

$$\begin{vmatrix} 4-\lambda & 1 & 0 \\ 1 & 1-\lambda & -1 \\ 0 & 3 & 4-\lambda \end{vmatrix} = 0,$$

which eventually leads to $\lambda = 2, 3, 4$. The respective eigenvectors are,

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Q2. Now we apply the eigenvalue theory to solving ODE systems. Solve:

(a) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ y(0) = 0.$$

(b) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ y(0) = 0.$$

(c) Question 4 on the first Laplace Transform sheet.

(d) $$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{subject to} \quad x(0) = 2,\ y(0) = 2,\ z(0) = 3.$$

In cases (a) and (d) you will be able to use the results of part of Q1 to lighten your load!

ANSWERS:

(a) The matrix here is the same as the matrix, A, in Q1. Noting that this system of equations is linear and has constant coefficients, we may set $(x, y) = e^{\lambda t}(X, Y)$, where X and Y are constants, and where λ is to be found. This results in precisely the expression we obtained earlier for the eigenvalues of A. Therefore we may use this information and write the present solution in the form,

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The given initial conditions are,

$$\begin{matrix} x(0) = 1 \\ \\ y(0) = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} A + B = 1 \\ \\ A - B = 0 \end{matrix} \quad \Rightarrow \quad A = B = \tfrac{1}{2}.$$

hence the final solution is,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{2}e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2}e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) Of similar difficulty to part (a). The values of λ are -1 and -6, and the corresponding eigenvectors are $(3, 1)^T$ and $(2, -1)^T$. Hence the general solution of the system is,

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + Be^{-6t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Application of the initial conditions gives $A = B = 1/5$. Hence the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t}\begin{pmatrix} 3/5 \\ 1/5 \end{pmatrix} + e^{-6t}\begin{pmatrix} 2/5 \\ -1/5 \end{pmatrix}.$$

(c) The solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + e^{-7t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

This is exactly the same solution as was given in the solutions to sheet 4, but it has been recast here in vector form.

(d) This matrix is C in Q1. The eigenvalues are $\lambda = 1, 2, 3$ and the respective eigenvectors are,

$$\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

These are in a form which is not normalised because the context of an ODE solution means that they are multiplied by an arbitrary constant. Hence we may say that the general solution for this system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ae^{t} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + Be^{2t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + Ce^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the problem stated, we have the initial conditions that $x(0) = 2$, $y(0) = 2$ and $z(0) = 3$. Substitution of this into the general solution yields,

$$\begin{aligned} -A + B + C &= 2 \\ A + B + C &= 2 \\ 3A + 2B + C &= 3 \end{aligned}$$

This may be written in matrix/vector form,

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix},$$

which may be solved using Gaussian Elimination. Hence $A = 0$, $B = 1$ and $C = 1$. The final solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Q3. Solve the following two systems of equations. Some of the work of Q2a may be used.

(a) $$\frac{d^2}{dt^2} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = 0.$$

(b) $$\frac{d^2}{dt^2} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = 0.$$

ANSWER:

(a) Let us run this independently from the answer to Q2a. This is a fourth order linear system composed of two second order ODEs. Therefore we expect four independent solutions. We proceed in the normal way, i.e. to substitute $x(t) = Xe^{\lambda t}$ and $y(t) = Ye^{\lambda t}$ where both X and Y are constants. Given our experience of this sort of thing we expect the exponentials to cancel, leaving us with

$$\lambda^2 \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

or

$$\begin{pmatrix} 2-\lambda^2 & -1 \\ -1 & 2-\lambda^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus λ^2 now plays the role of eigenvalue, and, given the results of Q1a, they are 1 and 3. Therefore there are four values for λ, namely, ± 1 and $\pm\sqrt{3}$. The eigenvectors corresponding to $\lambda^2 = 1$ and $\lambda^2 = 3$ are the same as those found for λ in Q1a and used in Q2a. Therefore, for this system we may write the general solution as,

$$\begin{pmatrix} x \\ y \end{pmatrix} = (Ae^{t} + Be^{-t}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (Ce^{\sqrt{3}t} + De^{-\sqrt{3}t}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So this new type of problem has two eigenvectors corresponding to the two different values of the eigenvalue, λ^2, but it also has four independent solutions and four arbitrary constants because the system we are solving is of fourth order.

We'll apply the boundary conditions in turn.

$$x(0) = 1 \quad \Rightarrow \quad 1 = A + B + C + D, \tag{1}$$

$$y(0) = 0 \quad \Rightarrow \quad 0 = A + B - C - D, \tag{2}$$

$$x'(0) = 0 \quad \Rightarrow \quad 0 = A - B + \sqrt{3}(C - D), \tag{3}$$

$$y'(0) = 0 \quad \Rightarrow \quad 0 = A - B - \sqrt{3}(C - D). \tag{4}$$

Astonishingly this set of four simultaneous equations in four unknowns may be solved easily. Eqs. (1) and (2) yield,

$$A + B = \tfrac{1}{2}, \qquad C + D = \tfrac{1}{2},$$

while equations (3) and (4) yield,

$$A - B = 0, \qquad C - D = 0.$$

From this we find that,

$$A = B = C = D = \tfrac{1}{4},$$

and hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{4}(e^t + e^{-t}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{4}(e^{\sqrt{3}t} + e^{-\sqrt{3}t}) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

or, more compactly, as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{2}\cosh t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2}\cosh \sqrt{3}t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) In this case we get $\lambda^2 = -1, -3$ and hence $\lambda = \pm j, \pm\sqrt{3}j$. Note that these aren't complex eigenvalues since λ^2 is real and it is λ^2 which is the eigenvalue. The general solutions may be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = (Ae^{jt} + Be^{-jt}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (Ce^{\sqrt{3}jt} + De^{-\sqrt{3}jt}) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

or equivalently as,

$$\begin{pmatrix} x \\ y \end{pmatrix} = (A\cos t + B\sin t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (C\sin\sqrt{3}t + D\sin\sqrt{3}t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Given the solution to part (e), it may come as no surprise that the final solution here, once the initial conditions have been used, is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{2}\cos t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2}\cos\sqrt{3}t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Just to fix ideas a little more, refer back to the solution of Q2d, which is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ae^t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + Be^{2t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + Ce^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where the three vectors shown here are the eigenvalues. If we were to replace d/dt by d^2/dt^2 in the governing equations for which this is the solution then the general solution of this new sixth order system would be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (Ae^t + Be^{-t}) \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + (Ce^{\sqrt{2}t} + De^{-\sqrt{2}t}) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + (Ee^{\sqrt{3}t} + Fe^{-\sqrt{3}t}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Q4. Solve the following systems of ODEs.

$$\text{(a)}\quad \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{(b)}\quad \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{(c)}\quad \frac{d^2}{dt^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If you have already found the eigenvalues and eigenvectors of J in Q1, then this should be a very quick question to answer.

ANSWER:

(a) Given that this is the same matrix as J in Q1, we may write down the solutions of this ODE in the form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ae^{2t}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + Ce^{4t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) This matrix is no longer J but $-J$. Its eigenvalues will be the negative of those of J but the corresponding eigenvectors will be the same. Hence the solution is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ae^{-2t}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + Ce^{-4t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Again the matrix is J but the equations are of second order. Hence we have $\lambda^2 = 2, 3, 4$. The solution of this sixth order system is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t})\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + (Ce^{\sqrt{3}t} + De^{-\sqrt{3}t})\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + (Ee^{2t} + Fe^{-2t})\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Q5. One of the applications of matrix theory is in the area of Markov chains, which is about probabilities that are associated with sequences of events. This is an example adapted from the textbook by Glyn James.

It is stated that dry days follow dry days with a probability of 0.5 while wet days follow wet days with probability, 0.6. The notation, $P(D_n)$, is the probability that day n is dry, and clearly $P(W_n) = 1 - P(D_n)$ is that it is wet on the same day, where no other choices of weather are available! All of this may be written as follows,

$$\begin{pmatrix} P(D_{n+1}) \\ P(W_{n+1}) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix}\begin{pmatrix} P(D_n) \\ P(W_n) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} P(D_0) \\ P(W_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here, the matrix is called a probability transition matrix and the initial condition (it is certainly dry!) is given. Although not necessary, see if you can understand how the matrix/vector equation has been constructed.

(a) Given the initial condition, find $P(D_1)$ and $P(W_1)$ using $n = 0$ in the above equation to find the prediction for the weather on day 1. Carry on like this to, say, day 4 to see if you can guess what the long-term trend is.

(b) Now find the eigenvalues and eigenvectors of the probability transition matrix, and see if you can relate these to your predicted long-term trend found in part (a).

(c) Given that $A\underline{v} = \lambda\underline{v}$ for eigenvectors and eigenvalues, the following is derived,

$$A^2\underline{v} = A(A\underline{v}) = A(\lambda\underline{v}) = \lambda\, A\underline{v} = \lambda^2\underline{v},$$

and so on for A^3, A^4 and so on. Now rewrite the original initial condition in the form,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\underline{v}_1 + B\underline{v}_2,$$

i.e. find the constants A and B; here $\underline{v}_1$ and $\underline{v}_2$ are the eigenvectors. Now find out what happens as successive days fly by, but always keep the result in terms of a sum of multiples of the eigenvectors.

(d) The property you have just uncovered is a feature of probability transition matrices: one eigenvalue is equal to 1 and the corresponding eigenvector is the long-term trend. All of the other eigenvalues are smaller in magnitude. This happens because the elements in each column of the probability transition matrix adds to 1. Check all of these statements by finding the eigenvalues and eigenvectors of

$$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}.$$

(e) Finally we wish to design our probability so that we can have dry weekends and wet weekdays, so the long-term behaviour is that it will be dry 2/7ths of the time and wet 5/7ths of the time. Let $a = 0.5$ (as it was at the start) and find the value of b which will ensure this outcome.

ANSWER: We are given,

$$\begin{pmatrix} P(D_{n+1}) \\ P(W_{n+1}) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} P(D_n) \\ P(W_n) \end{pmatrix}.$$

Generally such a pair of equations would be derived from a tree diagram of some sort. But as we will find out, if today is dry then tomorrow is characterised by,

$$\begin{pmatrix} P(D_1) \\ P(W_1) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

which fulfils the statement in the first part of the question that a dry day follows a dry day with a probability of 0.5. Clearly it will also be wet with the same probability, and this justifies where the second 0.5 is in the matrix. Likewise, if today is wet then,

$$\begin{pmatrix} P(D_1) \\ P(W_1) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$$

and so a wet day follows a wet day with probability 0.6.

(a) Now we have to find a sequence of probabilities for the first few days.

Day 1:
$$\begin{pmatrix} P(D_1) \\ P(W_1) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

Day 2:
$$\begin{pmatrix} P(D_2) \\ P(W_2) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} P(D_1) \\ P(W_1) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix}$$

Day 3:
$$\begin{pmatrix} P(D_3) \\ P(W_3) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} P(D_2) \\ P(W_2) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix} = \begin{pmatrix} 0.445 \\ 0.555 \end{pmatrix}$$

Day 4:
$$\begin{pmatrix} P(D_4) \\ P(W_4) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} P(D_3) \\ P(W_3) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 0.445 \\ 0.555 \end{pmatrix} = \begin{pmatrix} 0.4445 \\ 0.5555 \end{pmatrix}$$

So this appears to be heading towards,

$$\begin{pmatrix} P(D_\infty) \\ P(W_\infty) \end{pmatrix} = \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix}.$$

One thought is that the daily evolution of the probability vector shows how quickly the memory of that original dry day has faded!

(b) We need to find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} 0 &= \begin{vmatrix} 0.5-\lambda & 0.4 \\ 0.5 & 0.6-\lambda \end{vmatrix} \\ &= (0.5-\lambda)(0.6-\lambda) - 0.2 \\ &= \lambda^2 - 1.1\lambda + 0.1 \\ &= (\lambda - 1)(\lambda - 0.1). \end{aligned}$$

Hence the eigenvalues are $\lambda = 0.1$ and $\lambda = 1$.

In general an eigenvector will satisfy,

$$\begin{pmatrix} 0.5-\lambda & 0.4 \\ 0.5 & 0.6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 0.1$ we have,

$$\begin{pmatrix} 0.4 & 0.4 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

When $\lambda = 1$ we have

$$\begin{pmatrix} -0.5 & 0.4 \\ 0.5 & -0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Normally we could leave these eigenvectors as derived above. However, this has been undertaken within the context of probability. The components of the first eigenvector add to zero, and therefore this one may be left as it is. On the other hand, the components of the second eigenvector must add to 1 because the two components together must account for everything that might happen weather-wise, i.e. it will be either wet or dry and therefore the sum must be 1. Therefore we will use

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix}$$

as the eigenvector.

The eigenvector corresponding to $\lambda = 1$ is precisely the same as the long-term trend we computed earlier. Indeed, this may be seen because,

$$\begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} = \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix}.$$

I also note that

$$\begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.1 \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

as may be expected since $(1, -1)$ is the eigenvector corresponding to $\lambda = 0.1$.

(c) The eigenvectors are $\begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and therefore we may write the initial condition as,

$$\underline{v}_0 \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + 5/9 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;=\; \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + \begin{pmatrix} 5/9 \\ -5/9 \end{pmatrix}.$$

I have written the probability vector for day 0 as $\underline{v}_0$. So for day 1 we have,

$$\underline{v}_1 \;=\; M\underline{v}_0 \;=\; M\begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + M\begin{pmatrix} 5/9 \\ -5/9 \end{pmatrix} \;=\; \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + 0.1\begin{pmatrix} 5/9 \\ -5/9 \end{pmatrix}.$$

For days 2 and 3 we obtain,

$$\underline{v}_2 \;=\; M\underline{v}_1 \;=\; M\begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + 0.1M\begin{pmatrix} 5/9 \\ -5/9 \end{pmatrix} \;=\; \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + 0.01\begin{pmatrix} 5/9 \\ -5/9 \end{pmatrix},$$

and

$$\underline{v}_3 \;=\; M\underline{v}_2 \;=\; M\begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + 0.01M\begin{pmatrix} 5/9 \\ -5/9 \end{pmatrix} \;=\; \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + 0.001\begin{pmatrix} 5/9 \\ -bbb59 \end{pmatrix}.$$

Clearly the expression for day n is,

$$\underline{v}_n = \begin{pmatrix} 4/9 \\ 5/9 \end{pmatrix} + 10^{-n}\begin{pmatrix} 5/9 \\ -5/9 \end{pmatrix},$$

and therefore the reason why the original sequence of probability vectors evolved so quickly towards the long-term state is that the transient part $((1,-1))$ reduces by a factor of 10 each day.

Note: A more general observation is that successive premultiplications of a vector by a matrix will eventually yield the eigenvector corresponding to the eigenvalue of largest magnitude. If, in addition, the vector is normalised before each multiplication, then the number by which the vector is divided when normalised is that eigenvalue. If we take the matrix,

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

as an example, and we let the initial vector be,

$$\underline{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then we define subsequent vectors in the following way:

(i) Compute $\underline{\xi} = M\underline{v}_n$; (ii) let $\alpha = |\underline{\xi}|$; (iii) let $\underline{v}_{n+1} = (1/\alpha)\underline{\xi}$.

So α should tend towards the largest eigenvalue and $\underline{v}$ to its eigenvector. We'll try it out:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0.970143 \\ 0.242536 \end{pmatrix} \to \begin{pmatrix} 0.953583 \\ 0.301131 \end{pmatrix} \to \begin{pmatrix} 0.949689 \\ 0.313195 \end{pmatrix} \to \begin{pmatrix} 0.948885 \\ 0.315621 \end{pmatrix} \to \begin{pmatrix} 0.948724 \\ 0.316106 \end{pmatrix} \to$$

$$\alpha = 4.123106 \qquad \alpha = 4.832488 \qquad \alpha = 4.967654 \qquad \alpha = 4.993586 \qquad \alpha = 4.998719$$

We can guess that $\alpha \to 5$, and this is correct. Indeed this matrix has the eigenvalues, $\lambda = 5, 1$, and the corresponding normalised eigenvectors:

$$\begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} = \begin{pmatrix} 0.948683 \\ 0.316228 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0.707107 \\ -0.707107 \end{pmatrix}.$$

Finally (for this note, that is) this is an example of the Power Method (in effect it is the behaviour of successive powers of the matrix). The speed of convergence depends on the relative magnitudes of the largest two eigenvalue amplitudes, and I chose a case where convergence is quite quick. In "real life" the eigenvalue with the smallest magnitude is the one which is required, and this is equivalent to multiplication by the inverse matrix. This is called the Inverse Power Method.

(d) We'll find the eigenvalues and eigenvectors of

$$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}.$$

Therefore we need to solve,

$$0 \;=\; \begin{vmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{vmatrix} \;=\; \lambda^2-(a+1-b)\lambda+a-b \;=\; (\lambda-1)(\lambda-(a-b)),$$

and hence $\lambda = 1, a-b$. The eigenvectors are,

$$\lambda = 1: \qquad \frac{1}{b+1-a}\begin{pmatrix} b \\ 1-a \end{pmatrix} \qquad\qquad \lambda = a-b: \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Note that I have 'normalised' the eigenvector for $\lambda = 1$ by making sure that its entries add to 1, as we did before.

Therefore a matrix of this form clearly has one eigenvalue which is equal to 1.

(e) The final task is to obtain a $2/7$ long-term probability of it being dry when $a = 0.5$ has been taken. This means that the first entry in the eigenvector for $\lambda = 1$ must be equal to $2/7$. Hence,

$$\tfrac{2}{7} = \frac{b}{b+1-a} = \frac{b}{b+0.5} \quad\Rightarrow\quad \tfrac{2}{7}(b+\tfrac{1}{2}) = b \quad\Rightarrow\quad b = 0.2.$$

Therefore the required probability transition martrix is,

$$\begin{pmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{pmatrix}.$$

Checking:

$$\begin{pmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{pmatrix}\begin{pmatrix} 2/7 \\ 5/7 \end{pmatrix} = \begin{pmatrix} 2/7 \\ 5/7 \end{pmatrix}.$$

Q6. This final question is lengthy and well above the standard required for the exam. However, it may be done if one is led carefully through it. The aim is to find the eigenvalues of tridiagonal matrices such as the following:

$$J_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad J_3 = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad J_4 = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

You have already met these matrices in the problem sheet on determinants.

(i) Find the eigenvalues of J_2, J_3, J_4 and J_5 by using the standard techniques. I am not interested in finding the eigenvectors for now. Note that the factor $(\lambda+2)$ plays a vital role, and therefore *do not* expand your expressions for the determinants; keep them in terms of powers of $(\lambda+2)$. You should be able to find simple analytical values for the eigenvalues, even for J_5 (for which three of the five eigenvalues are negative integers).

(ii) While undertaking part (i), you should have noticed how $\det(J_n)$ may be written in terms of $\det(J_{n-1})$ and $\det(J_{n-2})$ in the same manner as we found in Q6 of the determinants problem sheet. Show that,

$$\det(J_n) = -(2+\lambda)\det(J_{n-1}) - \det(J_{n-2}).$$

Now use your expressions for $\det(J_2)$ and $\det(J_3)$ obtained in part (i) to show that your expression for $\det(J_4)$ is correct. Likewise show that your expression for $\det(J_5)$ is correct. What is the polynomial which represents $\det(J_6) = 0$?

(iii) Now we will go into further detail with J_5 to find a general way of finding the eigenvalues for J_n and the eigenvectors. Assume that the eigenvector for J_5 has the following form,

$$\begin{pmatrix} -2-\lambda & 1 & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & 0 & 0 \\ 0 & 1 & -2-\lambda & 1 & 0 \\ 0 & 0 & 1 & -2-\lambda & 1 \\ 0 & 0 & 0 & 1 & -2-\lambda \end{pmatrix} \begin{pmatrix} \sin\alpha \\ \sin 2\alpha \\ \sin 3\alpha \\ \sin 4\alpha \\ \sin 5\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where α is currently unknown. Now write out the equation corresponding to, say, row 3 of this equation, rewrite this equation using multiple angle formulae (i.e. let $\sin 4\alpha = \sin(3\alpha + \alpha)$ and $\sin 2\alpha = \sin(3\alpha - \alpha)$), and hence show that,

$$\lambda = -2 + 2\cos\alpha.$$

Row 1 of the matrix equation is a special case as it has only two coefficients; check that it too gives the same expression for λ. Row 5 is also a special case, but this should yield $\sin 6\alpha = 0$. Now you are in a position to write down a simple formula for the eigenvalues, λ, for J_5 which should fit with your original calculations. What are the eigenvectors? What is the implication for J_n in general?

ANSWER:

Now, when I said that this is lengthy and over and above what is intended for the unit, I certainly meant it. However, nothing which appears below is technically beyond what I presented in the lectures. Granted, the guess for the eigenvector would appear to be a little out-of-the-blue, but if you had determined the eigenvectors for J_4 and J_5 and then plotted them (in the manner I used in the lectures), then it might not have been such a bizarre idea. Indeed, the shapes of curves can be a useful guide to possible analytical solutions.

(i) For J_2 we have,

$$0 = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1.$$

Hence $\lambda = -2 \pm 1 = -1, -3$.

For J_3 we have,

$$0 = \begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= -(2+\lambda)\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & -2-\lambda \end{vmatrix}$$

$$= -(2+\lambda)\Big[(2+\lambda)^2 - 2\Big].$$

Hence, $\lambda = -2, -2 \pm \sqrt{3}$. For later purposes, the polynomial for λ may also be written in the form,

$$\det(J_3) = 2(2+\lambda) - (2+\lambda)^3 = 0,$$

while that for J_2 is,

$$\det(J_2) = (2+\lambda)^2 - 1 = 0.$$

For J_4 we have,

$$0 = \begin{vmatrix} -2-\lambda & 1 & 0 & 0 \\ 1 & -2-\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 1 \\ 0 & 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= -(2+\lambda)\begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= -(2+\lambda)\det(J_3) - \det(J_2)$$

$$= (2+\lambda)^4 - 3(2+\lambda)^2 + 1.$$

The penultimate line was put in just to show how the general case might be found, but the final polynomial for λ is given below it. One may now solve the quadratic for $(2+\lambda)^2$ to find,

$$(2+\lambda)^2 = \frac{3 \pm \sqrt{5}}{2},$$

and hence,

$$\lambda = -2 \pm \sqrt{\frac{3 \pm \sqrt{5}}{2}},$$

where all four choices of sign are needed. We also have,

$$\det(J_4) = (2+\lambda)^4 - 3(2+\lambda)^2 + 1.$$

For J_5 we have,

$$0 = \begin{vmatrix} -2-\lambda & 1 & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & 0 & 0 \\ 0 & 1 & -2-\lambda & 1 & 0 \\ 0 & 0 & 1 & -2-\lambda & 1 \\ 0 & 0 & 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= -(2+\lambda)\begin{vmatrix} -2-\lambda & 1 & 0 & 0 \\ 1 & -2-\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 1 \\ 0 & 0 & 1 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -2-\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 1 \\ 0 & 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= -(2+\lambda)\det(J_4) - \det(J_3)$$

$$= -(2+\lambda)\Big[(2+\lambda)^4 - 4(2+\lambda)^2 + 3\Big]$$

$$= -(2+\lambda)\Big[(2+\lambda)^2 - 3\Big]\Big[(2+\lambda)^2 - 1\Big].$$

Fortunately we are able to factorise the quintic quite easily (because we left all of the instances of $(\lambda+2)$ intact rather than multiplying them out), and hence we get,

$$\lambda = -3, -2, -1, -2 \pm \sqrt{3}.$$

We also have,

$$\det(J_5) = -(2+\lambda)^5 + 4(2+\lambda)^3 - 3(2+\lambda).$$

(ii) Given the way that the solutions to part (i) were written, it is clear that

$$\det(J_n) = -(2+\lambda)\det(J_{n-1}) - \det(J_{n-2})$$

is true in general. It is also straightforward to verify that our explicit expressions for the determinants of J_n satisfy this formula.

We may use the formula to show that,

$$\det(J_6) = (2+\lambda)^6 - 5(2+\lambda)^4 + 6(2+\lambda)^2 - 1.$$

(iii) If we multiply out row 3 of the given matrix/vector system, then we get,

$$\sin 2\alpha - (2+\lambda)\sin 3\alpha + \sin 4\alpha = 0.$$

If we use the given multiple angle formulae, then this expression becomes

$$\Big[\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha\Big] - (2+\lambda)\sin 3\alpha + \Big[\sin 3\alpha\cos\alpha + \cos 3\alpha\sin\alpha\Big] = 0$$

$$\Rightarrow \qquad \sin 3\alpha\Big[2\cos\alpha - 2 - \lambda\Big] = 0, \qquad (1)$$

$$\Rightarrow \qquad \lambda = -2 + 2\cos\alpha.$$

If we had multiplied out row 2 of the matrix, then Eq. (1) would have been obtained again, but the term in square brackets would have been multiplied by $\sin 2\alpha$. Likewise $\sin 4\alpha$ for row 4. Multiplying out row 1 yields,

$$-(2+\lambda)\sin\alpha + \sin 2\alpha = 0,$$

which is equivalent to,

$$\sin\alpha\Big[2\cos\alpha - 2 - \lambda\Big] = 0,$$

and therefore $\lambda = -2 + 2\cos\alpha$, again.

When we multiply out row 5, we get,

$$-(2+\lambda)\sin 5\alpha + \sin 4\alpha = 0$$

$$\Rightarrow \qquad -(2+\lambda)\sin 5\alpha + \sin(5\alpha - \alpha) = 0$$

$$\Rightarrow \qquad -(2+\lambda)\sin 5\alpha + \sin 5\alpha\cos\alpha - \cos 5\alpha\sin\alpha = 0$$

$$\Rightarrow \qquad \sin 5\alpha\cos\alpha + \cos 5\alpha\sin\alpha = 0$$

on using the fact that $\lambda = -2 + 2\cos\alpha$,

$$\sin 6\alpha = 0.$$

This means that 6α must be a multiple of π. If we let $\alpha = \pi/6$, then

$$\lambda = -2 + 2\cos(i\pi/6), \qquad i = 1, 2, 3, 4, 5.$$

Evaluation of these five different values of λ will yield the five we found earlier, namely, $-2-\sqrt{3}$, -3, -2, -1, $-2+\sqrt{3}$, in that order.

If we take $i = 1$, which is the eigenvalue of smallest magnitude $(-2+\sqrt{3})$, then the eigenvector is

$$\begin{pmatrix} \sin(\pi/6) \\ \sin(2\pi/6) \\ \sin(3\pi/6) \\ \sin(4\pi/6) \\ \sin(5\pi/6) \end{pmatrix}.$$

Plotting this out will show that these values are sampled points on half a sine wave.

If we take $i = 2$, for which $\lambda = -1$, we have the eigenvector,

$$\begin{pmatrix} \sin(2\pi/6) \\ \sin(4\pi/6) \\ \sin(6\pi/6) \\ \sin(8\pi/6) \\ \sin(10\pi/6) \end{pmatrix},$$

which are sampled points on a full sine wave.

For eigenvalue n, then the eigenvector is

$$\begin{pmatrix} \sin(n\pi/6) \\ \sin(2n\pi/6) \\ \sin(3n\pi/6) \\ \sin(4n\pi/6) \\ \sin(5n\pi/6) \end{pmatrix}.$$

If we have a matrix with $N-1$ rows and columns but which has the same structure, then an identical analysis will show that $\sin N\alpha = 0$, and therefore that $N\alpha$ is a multiple of π. Therefore the n^{th} eigenvalue is $\lambda = -2 + 2\cos(n\pi/N)$, and the corresponding eigenvector is

$$\begin{pmatrix} \sin(n\pi/N) \\ \sin(2n\pi/N) \\ \sin(3n\pi/N) \\ \vdots \\ \sin((N-2)n\pi/N) \\ \sin((N-1)n\pi/N) \end{pmatrix}.$$

And as a final thought, it is clear that all the eigenvalues of J_n, whatever the value of n, must lie between zero and -4 because $-4 < -2 + 2\cos(n\pi/N) < 0$ is always true.