University of Bath, Department of Mechanical Engineering

ME10305 Mathematics 2.

Matrices Sheet 2 — Determinants, Cramer's Rule and Gaussian Elimination.

Q1. Find the determinant of the following matrices. Which matrices are singular (i.e. have a zero determinant)? For (d) and (g) attempt the evaluation of the determinant in more than one way just to practice the skill.

$$\text{(a)}\ \begin{pmatrix} 6 & 2 \\ 8 & 3 \end{pmatrix} \qquad \text{(b)}\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(c)}\ \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \qquad \text{(d)}\ \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 1 & 2 & -1 \end{pmatrix}$$

$$\text{(e)}\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \text{(f)}\ \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \qquad \text{(g)}\ \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & -3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{(h)}\ \begin{pmatrix} b & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{pmatrix}$$

ANSWER:

(*a*)
$$\begin{vmatrix} 6 & 2 \\ 8 & 3 \end{vmatrix} = (6 \times 3) - (8 \times 2) = 18 - 16 = 2.$$

(*b*)
$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

(*c*)
$$\begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = (4 \times 1) - (-2) \times (-2) = 4 - 4 = 0.$$

(*d*)
$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 1 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} 3 & -3 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} = 0$$

In this case we expanded abut the first column in order to expoint the zero. An alternative approach would be to add the third column to the second one to obtain,

$$\begin{vmatrix} 2 & 2 & 1 \\ 0 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = -(-3)\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0.$$

(e) We may take the brute-force approach, as follows,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = (45 - 48) - 2(36 - 42) + 3(32 - 35) = 0,$$

where we have expanded about the first row, or we may use row manipulations, as follows,

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} \quad \begin{matrix} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{matrix} \\ &= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} \quad \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} \\ &= 0 \qquad \text{on expanding about the third row.} \end{aligned}$$

(*f*)
$$\begin{vmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{vmatrix} = 1\left[\tfrac{1}{3}\cdot\tfrac{1}{5} - \tfrac{1}{4}\cdot\tfrac{1}{4}\right] - \tfrac{1}{2}\left[\tfrac{1}{2}\cdot\tfrac{1}{5} - \tfrac{1}{3}\cdot\tfrac{1}{4}\right] + \tfrac{1}{3}\left[\tfrac{1}{2}\cdot\tfrac{1}{4} - \tfrac{1}{3}\cdot\tfrac{1}{3}\right] = 1/2160$$

It is possible to use row and/or column manipulations for this, but the number of fractions becomes difficult to handle.

(g) Again, the brute force approach (expanding about the 1st column) gives,

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & -3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & -3 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 & 1 \\ 3 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 3 & -3 & 1 \\ 2 & -1 & 0 \end{vmatrix} = 2[3+3] + 0 - 1[3+3] = 6.$$

Using row manipulations gives,

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & -3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} \qquad R_1 - R_4$$

$$= \begin{vmatrix} 3 & -3 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \qquad \text{expanding about } C_1$$

$$= \begin{vmatrix} 2 & -4 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad C_1^{\text{new}} = C_1^{\text{old}} - C_3, \quad C_2^{\text{new}} = C_2^{\text{old}} - C_3$$

$$= \begin{vmatrix} 2 & -4 \\ 2 & -1 \end{vmatrix} \qquad \text{expanding about } R_3$$

$$= 6$$

Note that there are many different ways of using row and column manipulations to get this result.

(h) One way is to subtract row 1 from row 2, row 1 from row 3, and row 1 from row 4, followed by adding each of columns 2, 3 and 4 to column 1. This yields the determinant of the following matrix:

$$\begin{vmatrix} b+3a & a & a & a \\ 0 & b-a & 0 & 0 \\ 0 & 0 & b-a & 0 \\ 0 & 0 & 0 & b-a \end{vmatrix}$$

whose value is $(b+3a)(b-a)^3$.

Clearly this matrix has a zero determinant when $b = a$, and obviously so because all four rows of the original matrix are then identical. Much less obviously the matrix is singular when $b = -3a$.

Finally, we may say that matrices (c), (d) and (e) are singular because their determinants are zero.

Q2. The matrix J_n is an $n \times n$ matrix where the diagonal entries have the value -2, the superdiagonal and subdiagonal entries the value 1, and 0 elsewhere. For example, J_1, J_2 and J_5 are

$$J_1 = (\,-2\,) \qquad J_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \qquad J_5 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Such matrices arise in the numerical solution of second order ordinary differential equations.

Assume that $|J_1| = -2$, and then evaluate $|J_2|$, $|J_3|$, $|J_4|$ and $|J_5|$ directly from the matrix definitions. This should show you how to derive the recurrence relation,

$$|J_n| = -2|J_{n-1}| - |J_{n-2}|.$$

Finally, what is the explicit value of $|J_n|$?

ANSWER: We're given that $|J_1| = -2$, which may come as a bit of a surprise because the modulus signs here mean a "determinant" rather than an absolute value. So there is a difference between $|J_1|{=}|(-2)| = -2$ and $|-2| = 2$, the latter being the modulus of a number. This issue is exceptionally unlikely to arise in practice because there's no need to have a 1×1 matrix!

The following is easy:

$$|J_2| = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3.$$

For the next one we have,

$$|J_3| = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = -2\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = (-2)(3) - (1)(-2) = -4.$$

This one is a little longer:

$$\begin{aligned} |J_4| &= \begin{vmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{vmatrix} && \text{expand using the 1st row} \\ &= -2\begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} && \text{expand using the 1st column} \\ &= -2\begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ &= -2|J_3| - |J_2|. \end{aligned}$$

So we have shown that $|J_4| = -2|J_3| - |J_2|$. When one decides to do a similar expansion for $|J_5|$, it becomes apparent that the only change in the analysis from that of $|J_4|$ is the addition of an extra column on the right and an extra row at the bottom. Therefore this result does indeed represent the general case.

But just to check, consider the determinant of J_n, an $n \times n$ matrix.

$$|J_n| = \begin{vmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{vmatrix}$$

Expanding about the first row yields

$$|J_n| = -2\begin{vmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & & & & \\ 0 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{vmatrix}$$

The first determinant here is $|J_{n-1}|$ (noting that it must be an $(n-1) \times (n-1)$ determinant. If we expand the second determinant about the first column, we see that it is equal to $|J_{n-2}|$. Hence we obtain the result,

$$|J_n| = -2|J_{n-1}| - |J_{n-2}|.$$

We already have $|J_1| = -2$ and $|J_2| = 3$. Although we have already found that $|J_3| = -4$ directly, the rcurrence relation gives,

$$|J_3| = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = -2\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = (-2)(3) - (1)(-2) = -4.$$

Using the formula we obtain the following in turn: $|J_4| = 5$, $|J_5| = -6$, $|J_6| = 7$ and so on. Hence

$$|J_n| = (-1)^n(n+1).$$

Q3. Use Cramer's rule to solve the following systems of equations.

(a) $$\begin{aligned} 2x + 5y &= -1 \\ -3x + 2y &= 2 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 1 \\ 6x_1 - 2x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 3x_2 - x_3 &= 3 \\ x_2 - 7x_3 &= 2 \\ 2x_1 - 5x_3 &= 1 \end{aligned}$$

ANSWER: (a)

The equation is $\begin{pmatrix} 2 & 5 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. The determinant of the matrix is $\begin{vmatrix} 2 & 5 \\ -3 & 2 \end{vmatrix} = 4 + 15 = 19.$

Using Cramer's rule we have:

$$x = \tfrac{1}{19}\begin{vmatrix} -1 & 5 \\ 2 & 2 \end{vmatrix} = -\tfrac{12}{19} \qquad y = \tfrac{1}{19}\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} = \tfrac{1}{19}.$$

(b)
$$\begin{vmatrix} 2 & 3 & -2 \\ 6 & -2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -19$$

Using Cramer's rule:

$$x_1 = -\frac{1}{19}\begin{vmatrix} 1 & 3 & -2 \\ 2 & -2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \frac{-19}{-19} = 1.$$

$$x_2 = -\frac{1}{19}\begin{vmatrix} 2 & 1 & -2 \\ 6 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \frac{-19}{-19} = 1.$$

$$x_3 = -\frac{1}{19}\begin{vmatrix} 2 & 3 & 1 \\ 6 & -2 & 2 \\ 1 & -1 & 2 \end{vmatrix} = \frac{-38}{-19} = 2.$$

(c) The solution for this matrix is: $x_1 = 5/45$, $x_2 = 41/45$ and $x_3 = -7/45$.

Q4. Use Gaussian Elimination to solve the following systems of equations.

(a)
$$\begin{aligned} 2x + 5y &= -1 \\ -3x + 2y &= 2 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 1 \\ 6x_1 - 2x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 3x_2 - x_3 &= 3 \\ x_2 - 7x_3 &= 2 \\ 2x_1 - 5x_3 &= 1 \end{aligned}$$

Note that the above three systems of equations are identical to those in which were solved in Q3 using Cramer's rule.

(d)
$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 6 \\ 3 & 0 & 1 & 1 \\ -2 & 1 & -3 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \\ -1 \end{pmatrix}$$

(e)
$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 7 \end{pmatrix}$$

ANSWER: (a) Using Gaussian Elimination with the augmented matrix notation:

$$\begin{bmatrix} 2 & 5 & | & -1 \\ -3 & 2 & | & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 5 & | & -1 \\ 0 & \frac{19}{2} & | & \frac{1}{2} \end{bmatrix} \quad \begin{matrix} R_1 \\ R_2 + \frac{3}{2}R_1 \end{matrix}$$

Hence $2x + 5y = -1$ and $\frac{19}{2}y = \frac{1}{2}$. And therefore we obtain,

$$x = -12/19 \qquad y = 1/19.$$

(b) For the Gaussian Elimination algorithm, I have interchanged the order of the rows for convenience. Hence

$$\begin{bmatrix} 1 & -1 & 1 & | & 2 \\ 2 & 3 & -2 & | & 1 \\ 6 & -2 & -1 & | & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & | & 2 \\ 0 & 5 & -4 & | & -3 \\ 0 & 4 & -7 & | & -10 \end{bmatrix} \qquad \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 - 6\mathrm{R}_1 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 1 & | & 2 \\ 0 & 5 & -4 & | & -3 \\ 0 & 0 & -\frac{19}{5} & | & -\frac{38}{5} \end{bmatrix} \qquad \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 - \frac{4}{5}\mathrm{R}_2 \end{matrix}$$

Hence $x_3 = 2$, $x_2 = (-3 + 4x_3)/5 = 1$ and $x_1 = 2 - x_3 + x_2 = 1$.

(c) The solution for this matrix is: $x_1 = 19/48$, $x_2 = 41/48$ and $x_3 = -2/48$.

(d) Now a 4×4 system. It follows as above but takes a little longer.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 1 & 1 & -2 & 6 & | & -4 \\ 3 & 0 & 1 & 1 & | & 2 \\ -2 & 1 & -3 & 0 & | & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & -1 & 5 & | & -6 \\ 0 & -6 & 4 & -2 & | & -4 \\ 0 & 5 & -5 & 2 & | & 3 \end{bmatrix} \qquad \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 - 3\mathrm{R}_1 \\ \mathrm{R}_4 + 2\mathrm{R}_1 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & -1 & 5 & | & -6 \\ 0 & 0 & 10 & -32 & | & 32 \\ 0 & 0 & -10 & 27 & | & -27 \end{bmatrix} \qquad \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 - 6\mathrm{R}_2 \\ \mathrm{R}_4 + 5\mathrm{R}_2 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & -1 & 5 & | & -6 \\ 0 & 0 & 10 & -32 & | & 32 \\ 0 & 0 & 0 & -5 & | & 5 \end{bmatrix} \qquad \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4 + \mathrm{R}_3 \end{matrix}$$

From this we get d, c, b and a in turn. Hence,

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

(e) There are a lot of fractions with this Gaussian Elimination problem. If one wishes to avoid fractions, then it is possible to replace a row operation such as the new row 3 being equal to $\mathrm{R}_3 + \frac{2}{5}\mathrm{R}_1$, by the new row 3 being equal to $5\mathrm{R}_3 + 2\mathrm{R}_1$. This latter replacement is composed of adding twice the old row 1 to 5 times the old row 3 — this is perfectly valid since one is adding equations together.

The solution is,

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 3 \end{pmatrix}.$$

Q5. The matrix given in Q1e is singular, by which is meant that it has a zero determinant, and therefore it either has no solution or an infinite number of them. The aim of this question is to see how Gaussian Elimination copes with such a situation.

Try to solve the matrix/vector equation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

using Gaussian Elimination to see the manner in which the procedure fails when the matrix is singular. However, it is possible to write down solutions for this case. I haven't covered this in the lectures and therefore you'll need to work out how to do it.

Now try to find the solution when the right hand side vector is $(-2, 1, 5)^T$. Can you explain why two separate equations involving the same matrix has solutions in one case but not in another? (Hint: use $(a, b, c)^T$ as the right hand side as a third case.)

ANSWER: This problem involves solving two separate matrix/vector equations which use the same matrix. It is instructive to solve a third equation where the right hand side is $(a, b, c)^T$, which is why I have given the hint.

I will use the augmented matrix notation with all three right hand sides represented, so that it is not necessary to repeat the elimination part of the Gaussian Elimination algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -2 & a \\ 4 & 5 & 6 & 1 & 1 & b \\ 7 & 8 & 9 & 1 & 5 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -2 & a \\ 0 & -3 & -6 & -3 & 9 & b-4a \\ 0 & -6 & -12 & -6 & 19 & c-7a \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 - 4\mathrm{R}_1 \\ \mathrm{R}_3 - 7\mathrm{R}_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & -2 & a \\ 0 & -3 & -6 & -3 & 9 & b-4a \\ 0 & 0 & 0 & 0 & 1 & c-2b+a \end{array}\right] \begin{array}{l} \\ \\ \mathrm{R}_3 - 2\mathrm{R}_2 \end{array}$$

Note: all we have done here is to use the elimination part of Gaussian Elimination simultaneously for all three right hand vectors; this has been to avoid duplicating the elimination stage as it is applied to the matrix.

For the first right hand side the final row is equivalent to $0x + 0y + 0z = 0$ which is fine. This leaves us two equations in three unknowns. If we leave z as being unspecified and equal to α, say, then the second row of the matrix leads us to $y = 1 - 2\alpha$, and the first to $x = -1 + \alpha$. This solution may be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Therefore the first matrix/vector equation has an infinite number of solutions because α is an arbitrary number. In fact, the vector that is multiplying α is the eigenvector of the matrix which corresponds to the zero eigenvalue.

Consider now the second right hand side vector. The final row is equivalent to $0x + 0y + 0z = 1$, which does not have a solution. Therefore the second matrix/vector equation does not have a solution. This result is, in fact, typical.

For the third right hand side vector we see that if $c - 2b + a = 0$ we recover a situation where we can obtain an infinite number of solutions, whereas if $a - 2b + c \neq 0$ then there is no solution. Thus $c - 2b + a = 0$ is the condition whereby the equation has solutions, even though the matrix is singular.

Q6. The aim here is to find the inverse of some matrices using Gaussian Elimination starting with the identity matrix as part of the augmented matrix scheme. Other general properties of inverses will arise along the way. Treat this question as practice in Gaussian elimination; the computation of inverses takes too long in the examination context (with the possible exception of a tridiagonal 3×3 matrix). Find the inverses of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \end{pmatrix}$$

You may check your answer either by forming the product $M^{-1}M$ or the product MM^{-1} or by consulting the web page: `https://matrix.reshish.com/inverse.php`.

What conclusion can you draw about the inverses of matrices which are symmetric, antisymmetric or tridiagonal?

ANSWER

Case A. We employ the identity in the form of three vector-like right hand sides.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 - \mathrm{R}_1 \\ \mathrm{R}_3 - \mathrm{R}_1 \end{array}$$

Fortuitously, the act of making the terms that are in the 1st column and below the main diagonal equal to zero has also made it upper triangular. If we now find the vector solutions corresponding to each right hand side in turn (see the lecture notes) using back-substitution, then we eventually obtain,

$$A^{-1} = \tfrac{1}{6} \begin{pmatrix} 2 & 3 & 1 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

As already mentioned, for this particular matrix, it was somewhat unusual that the elimination of the terms, a_{21} and a_{31}, also gave $a_{32} = 0$. This saved some time but isn't to be expected in general.

For the remaining cases I won't give workings, for this takes up a lot of space, and indeed there are a few slightly different ways of doing the elimination.

$$B = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \begin{pmatrix} 5/6 & -2/3 & 1/6 \\ -1/6 & 1/3 & 1/6 \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$$

In this case the b_{21} was already zero.

$$C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad C^{-1} = \tfrac{1}{4} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

The matrix, C, is symmetric, and so is its inverse.

$$D = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \Rightarrow \quad D^{-1} = \frac{1}{5} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Again a symmetric matrix has a symmetric inverse. However, D is tridiagonal, but clearly its inverse is fully populated. The form of the inverse hints at some sort of pattern, so I thought that I would try the following:

$$\begin{pmatrix} 2 & -1 & . & . & . & . & . & . \\ -1 & 2 & -1 & . & . & . & . & . \\ . & -1 & 2 & -1 & . & . & . & . \\ . & . & -1 & 2 & -1 & . & . & . \\ . & . & . & -1 & 2 & -1 & . & . \\ . & . & . & . & -1 & 2 & -1 & . \\ . & . & . & . & . & -1 & 2 & -1 \\ . & . & . & . & . & . & -1 & 2 \end{pmatrix}^{-1} = \frac{1}{9} \begin{pmatrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 6 & 12 & 18 & 15 & 12 & 9 & 6 & 3 \\ 5 & 10 & 15 & 20 & 16 & 12 & 8 & 4 \\ 4 & 8 & 12 & 16 & 20 & 15 & 10 & 5 \\ 3 & 6 & 9 & 12 & 15 & 18 & 12 & 6 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

So yes, there is definitely a pattern which will allow us to predict the inverse of similarly-constructed larger matrices. These two tridiagonal matrices are also symmetric about the other diagonal, i.e. top left to bottom right, and the inverses also bear that extra symmetry.

$$E = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

This matrix, which is antisymmetric, does not have an inverse. Its determinant is zero, and therefore Cramer's rule (which we would apply if we had a set of equations to solve involving E) gives solutions with a zero denominator. Looking forward to the eigenvalue section of this topic, a zero determinant also means that at least one of the eigenvalues is zero.

$$F = \begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \end{pmatrix} \quad \Rightarrow \quad F^{-1} = \begin{pmatrix} 72 & -240 & 180 \\ -240 & 900 & -720 \\ 180 & -720 & 600 \end{pmatrix}$$

These are particularly large entries in the inverse. If we interpret the rows of F as being the normal directions of three planes, then it is clear that these three planes are close to being parallel. This suggests that the determinant of the matrix will be very small (I can say that because, should two of these planes be parallel, then the determinant will be zero). It turns out to be 1/43200.

D.A.S.R. 18/02/2021