University of Bath, Department of Mechanical Engineering

ME10305 Mathematics 2.

Matrices Sheet 1 — multiplication.

Q1. The matrices, A, B, C and D, are defined as follows,

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Classify all these matrices in terms of numbers of rows and columns. Now make a list of which pairs may be multiplied together (i.e. are *compatible* with respect to multiplication) — for example, both AB and BA belong to this list. Now find all the permissible products of two matrices.

ANSWER: The matrices are classified as follows:

$$A \text{ is } 3\times 2, \qquad B \text{ is } 2\times 3, \qquad C \text{ is } 2\times 2, \qquad D \text{ is } 3\times 3.$$

Given that we may only multiply matrices which are $n_1 \times n_2$ and $n_3 \times n_4$, in that order, when $n_2 = n_3$, i.e. the number of columns of the first matrix is equal to the number of rows of the second matrix, we have the following Table of possibilities:

AA	—	BA	2×2	CA	—	DA	3×2
AB	3×3	BB	—	CB	2×3	DB	—
AC	3×2	BC	—	CC	2×2	DC	—
AD	—	BD	2×3	CD	—	DD	3×3

The required products are

$$AB = \begin{pmatrix} 5 & -1 & 3 \\ 1 & -2 & 3 \\ 13 & -2 & 7 \end{pmatrix} \qquad AC = \begin{pmatrix} 0 & 3 \\ -3 & 3 \\ 1 & 7 \end{pmatrix} \qquad DA = \begin{pmatrix} 1 & 5 \\ 2 & 9 \\ 5 & 11 \end{pmatrix} \qquad DD = \begin{pmatrix} 5 & 4 & 1 \\ 4 & 6 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

$$BA = \begin{pmatrix} -3 & -2 \\ 9 & 13 \end{pmatrix} \qquad BD = \begin{pmatrix} 3 & 2 & -1 \\ 3 & 2 & 3 \end{pmatrix} \qquad CB = \begin{pmatrix} 0 & 3 & -4 \\ 3 & -3 & 5 \end{pmatrix} \qquad CC = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

Q2. Having now determined AB, where A and B are as given in Q1, write down $(AB)^T$, the transpose of AB. Now calculate B^TA^T. Is $(AB)^T = B^TA^T$? Is it obvious whether this last answer is true in general?

ANSWER: Given the above, we have,

$$(AB)^T = \begin{pmatrix} 5 & 1 & 13 \\ -1 & -2 & -2 \\ 3 & 3 & 7 \end{pmatrix}$$

We also have

$$B^TA^T = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 13 \\ -1 & -2 & -2 \\ 3 & 3 & 7 \end{pmatrix}.$$

Hence $(AB)^T = B^T A^T$ in this case. However, when is multiplying out the rows and columns for both AB and $B^T A^T$ it should have been quite clear that the same numbers were involved each time.

It is possible to show this in general in the following way, although it is a little mind-mangling. If I define a_{ij} to be entry of A which is in row i and column j, then the (i,j)–entry of $C = AB$ is given by,

$$c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj},$$

where N is the number of columns in A and the number of rows in B. Now we need to check out what happens if we set $D = B^T A^T$. First, it should be clear that if the (i,j)–entry in A is a_{ij}, then the (i,j) entry in A^T is a_{ji}, because the rows and columns have been interchanged. Hence the (i,j)–entry in $D = B^T A^T$ is given by,

$$d_{ij} = \sum_{k=1}^{N} b_{ki} a_{jk} = c_{ji}.$$

That was the mind-mangling bit — think about it! Hence $C = D^T$ and so $(AB)^T = B^T A^T$ in general.

Q3. The matrix A is defined by

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -3 \\ 1 & 2 & -1 \end{pmatrix}.$$

Find A^T. Now form the sums $A + A^T$ and $A - A^T$. What do you notice about these new matrices? Find the products AA^T and $A^T A$. What do you conclude from these results?

ANSWER: The transpose of A is given by

$$A^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 2 \\ 2 & -3 & -1 \end{pmatrix}.$$

Hence,

$$A + A^T = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 6 & -1 \\ 3 & -1 & -2 \end{pmatrix}, \qquad A - A^T = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -5 \\ -1 & 5 & 0 \end{pmatrix}.$$

Hence $A + A^T$ is a symmetric matrix, and $A - A^T$ is antisymmetric (not asymmetric). Note that an antisymmetric matrix always has zeroes on the leading diagonal.

We also get,

$$AA^T = \begin{pmatrix} 9 & -3 & 2 \\ -3 & 18 & 9 \\ 2 & 9 & 6 \end{pmatrix}, \qquad A^T A = \begin{pmatrix} 5 & 4 & 3 \\ 4 & 13 & -9 \\ 3 & -9 & 14 \end{pmatrix}.$$

Therefore these products are both symmetric matrices. However, $AA^T \neq A^T A$.

Q4. We have seen that matrix multiplication, where the matrices are compatible, yields $AB \neq BA$ in general, i.e. matrix multiplication is non-commutative. But I would like you to show that matrix multiplication is associative, that is, $A(BC) = (AB)C$, where the term in brackets is computed first. Check one specific case:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Now check the general case for 2×2 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Try to think of a way of generalising this result to any three square matrices, and then to any set of compatible matrices.

ANSWER: For the matrices given we do find that $A(BC) = (AB)C = \begin{pmatrix} 0 & 2 \\ 6 & 10 \end{pmatrix}$. For the general case we also find that associativity is satisfied. We obtain

$$A(BC) = (AB)C =$$

$$\begin{pmatrix} a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22} \end{pmatrix}.$$

The important thing to note about further generalisation is that the subscripts in each product of a, b and c in ABC follow a pattern. The first subscript of a and the second of c in each term gives where that product is situated in ABC. The other subscripts run through all the possible combinations allowable. So, if A, B and C were $n \times n$ matrices, we would have

$$ABC = \begin{pmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1i}b_{ij}c_{j1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1i}b_{ij}c_{j2} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1i}b_{ij}c_{jn} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2i}b_{ij}c_{j1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2i}b_{ij}c_{j2} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2i}b_{ij}c_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ni}b_{ij}c_{j1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ni}b_{ij}c_{j2} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ni}b_{ij}c_{jn} \end{pmatrix}.$$

Likewise, if A is $n \times m$, B is $m \times p$ and C is $p \times q$, we have

$$ABC = \begin{pmatrix} \sum_{i=1}^{m}\sum_{j=1}^{p} a_{1i}b_{ij}c_{j1} & \sum_{i=1}^{m}\sum_{j=1}^{p} a_{1i}b_{ij}c_{j2} & \cdots & \sum_{i=1}^{m}\sum_{j=1}^{p} a_{1i}b_{ij}c_{jq} \\ \sum_{i=1}^{m}\sum_{j=1}^{p} a_{2i}b_{ij}c_{j1} & \sum_{i=1}^{m}\sum_{j=1}^{p} a_{2i}b_{ij}c_{j2} & \cdots & \sum_{i=1}^{m}\sum_{j=1}^{p} a_{2i}b_{ij}c_{jq} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{m}\sum_{j=1}^{p} a_{ni}b_{ij}c_{j1} & \sum_{i=1}^{m}\sum_{j=1}^{p} a_{ni}b_{ij}c_{j2} & \cdots & \sum_{i=1}^{m}\sum_{j=1}^{p} a_{ni}b_{ij}c_{jq} \end{pmatrix}.$$

It is important to note that the general part of this question is beyond the remit of the course, and is not an aspect which will be examined.

Q5. Not really part of the syllabus, but I needed something to fill the gap at the bottom of this page! If one has two tridiagonal matrices which are compatible with respect to multiplication and are subsequently multiplied together, then is there is a general statement that can be made about the pattern of the components in that product?

ANSWER: Tridiagonal matrices are always square in shape, and therefore a compatible pair must be identical in shape. In the following I shall represent the general case using a pair of 6×6 general tridiagonal matrices:

$$\begin{pmatrix} \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \\ \bullet & \bullet & \bullet & \cdot & \cdot & \cdot \\ \cdot & \bullet & \bullet & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\ \cdot & \cdot & \cdot & \bullet & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet \end{pmatrix} \times \begin{pmatrix} \circ & \circ & \cdot & \cdot & \cdot & \cdot \\ \circ & \circ & \circ & \cdot & \cdot & \cdot \\ \cdot & \circ & \circ & \circ & \cdot & \cdot \\ \cdot & \cdot & \circ & \circ & \circ & \cdot \\ \cdot & \cdot & \cdot & \circ & \circ & \circ \\ \cdot & \cdot & \cdot & \cdot & \circ & \circ \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}.$$

In the above, dots represent zeros in the matrices which are being multiplied while the bullets and circles represent quantities that are very likely to be nonzero. The entries in the final matrix gives the count for how many bullet/circle multiplications are involved for each entry. It is clear that the pattern of nonzero entries is pentadiagonal. Hopefully it is quite clear that the same will be true for larger tridiagonal matrices.

Q6. Rotation matrices are important for many applications and are especially so in robotics. I am not going to teach this formally, but I would like to introduce them and to play around with them a little.

We may define the following three rotation matrices:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \qquad \text{(Rotation by an angle } \alpha \text{ about the } x\text{-axis.)}$$

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \qquad \text{(Rotation by an angle } \beta \text{ about the } y\text{-axis.)}$$

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(Rotation by an angle } \gamma \text{ about the } z\text{-axis.)}$$

Therefore if the position vector of a point is $\underline{r}$, and if that point is rotated by an angle, α, about the x-axis, then the new location of the point is given by the matrix/vector product, $R_x(\alpha)\,\underline{r}$. If this new point is subsequently rotated by γ about the z-axis, then its new location is given by, $R_z(\gamma)\,R_x(\alpha)\,\underline{r}$.

Thus a rotation about the x-axis followed by a rotation about the z-axis is $R_z(\gamma)\,R_x(\alpha)$, where the rotation matrices only appear to have been written down in the wrong order!

(i) Perhaps it is not surprising that the inverse matrix of $R_x(\alpha)$ is $R_x(-\alpha)$, given what this notation means. Check the $R_x(\alpha)\,R_x(-\alpha) = I$, the 3×3 identity matrix.

(ii) Find both $R_z(\gamma)\,R_x(\alpha)$ and $R_x(\alpha)\,R_z(\gamma)$. Are they equal? What is the implication of this general result? What about when $\alpha = \gamma = \frac{1}{4}\pi$? What about when $\alpha = \gamma = \frac{1}{2}\pi$?

(iii) If you really have time spare, then you could try the following. A point suffers the grave indignity of the following sequence of rotations: $R_x(\alpha)$ then $R_z(\gamma)$ then $R_x(-\alpha)$ then $R_z(-\gamma)$. This expresses a possibly naive thought that an arbitrarily chosen point will return to where it started after this sequence; do you think that it will? If not, what are the correct third and fourth rotations to cause the point to return?

ANSWER: (i) Given that $R_x(\alpha)$ is the rotation by an angle, α, about the x-axis, and that $R_x(-\alpha)$ is a rotation by an angle, $-\alpha$, then the latter merely undoes what the former did. Given that these are matrices, then they are mutual inverses. But we can check this using matrix multiplication:

$$R_x(-\alpha)R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) The products are as follows,

$$R_z(\gamma)R_x(\alpha) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\cos\alpha\sin\gamma & \sin\alpha\sin\gamma \\ \sin\gamma & \cos\alpha\cos\gamma & -\sin\alpha\cos\gamma \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

and

$$R_x(\alpha)R_z(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \cos\alpha\sin\gamma & \cos\alpha\cos\gamma & -\sin\alpha \\ \sin\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha \end{pmatrix}.$$

These are definitely two different matrices, although the respective terms on the main diagonal are identical. The practical implication is that one must specify the order in which two rotations are declared.

When $\alpha = \gamma = \frac{1}{4}\pi$, then,

$$R_z(\gamma)\,R_x(\alpha) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad R_x(\alpha)\,R_z(\gamma) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

So this isn't a special case.

When $\alpha = \gamma = \frac{1}{2}\pi$, then,

$$R_z(\gamma)\,R_x(\alpha) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad R_x(\alpha)\,R_z(\gamma) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

These matrices merely alter the coordinates, e.g.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ a \\ b \end{pmatrix}.$$

(iii) The suggested sequence of four successive rotations will not return the point to where it started.

$$R_z(-\gamma)\,R_x(-\alpha)\,R_z(\gamma)\,R_x(\alpha)\,\underline{r} \neq \underline{r}.$$

The first two rotations are,

$$R_z(\gamma)\,R_x(\alpha),$$

and the the sole way of getting an arbitrary point back to where it started is to unravel the rotations, first by undoing the z-rotation using $R_z(-\gamma)$, and then undoing the x-rotation using $R_x(-\alpha)$. Hence,

$$R_x(-\alpha)\,R_z(-\gamma)\,R_z(\gamma)\,R_x(\alpha)\,\underline{r} = \underline{r}.$$

Q7. A question for interest, perhaps. Fermat's last theorem is well-known: when a, b, c and n take positive integer values, the equation $a^n + b^n = c^n$ has solutions only when $n = 2$. However, a Michael Penn youtube video alerted me to the fact that this theorem doesn't apply when a, b and c are matrices! So here's a straightforward exercise in matrix multiplication to check if Prof. Penn is correct:

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^3 + \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}^3 = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}^3.$$

ANSWER: Yes, the multiplications work! We have,

$$A^3 = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix} \qquad B^3 = \begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \qquad C^3 = \begin{pmatrix} 0 & 9 \\ 3 & 0 \end{pmatrix}.$$

D.A.S.R. 25/02/2022